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A Geometric Summation of the Geometric Series

Let k be a finite field of size q. The n-dimensional projective space over k, $\mathbb{P}^n(k)$, is the set of orbits of $k^{n+1}\setminus\{0\}$ under the action of k^* given by scalar multiplication. Therefore $|\mathbb{P}^n(k)| = \frac{q^{n+1}-1}{q-1}$ since all orbits have the same size. Now, since the decomposition of $\mathbb{P}^n(k)$ into a disjoint union of affine spaces $\mathbb{P}^n(k) = A_0 \cup A_1 \cup \cdots \cup A_n$ where

$$A_i := \{(x_0 : \cdots : x_{i-1} : 1 : 0 : \cdots : 0) \,|\, x_j \in k\}$$

is in bijection with the affine space k^i, then $|\mathbb{P}^n(k)| = \sum_{i=0}^n q^i$. Thus for all n and power of primes q,

$$\sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1}.$$

Since this identity of univariate rational functions over $\mathbb{Q}$ is true for an infinite amount of numbers, it translates into an equality of rational functions. Hence

$$\sum_{i=0}^{n} X^i = \frac{X^{n+1}-1}{X-1}$$

with X indeterminate. Can this method be used to prove other identities by interpreting them geometrically using algebraic varieties over finite fields?

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