# Condition Numbers & Probability for Explaining Algorithms in Computational Geometry

Josué Tonelli-Cueto
UT San Antonio

MINDS & CIS Seminar Series
6/sep/2022

# Fun Fact of the Day (6/Sep/2022)

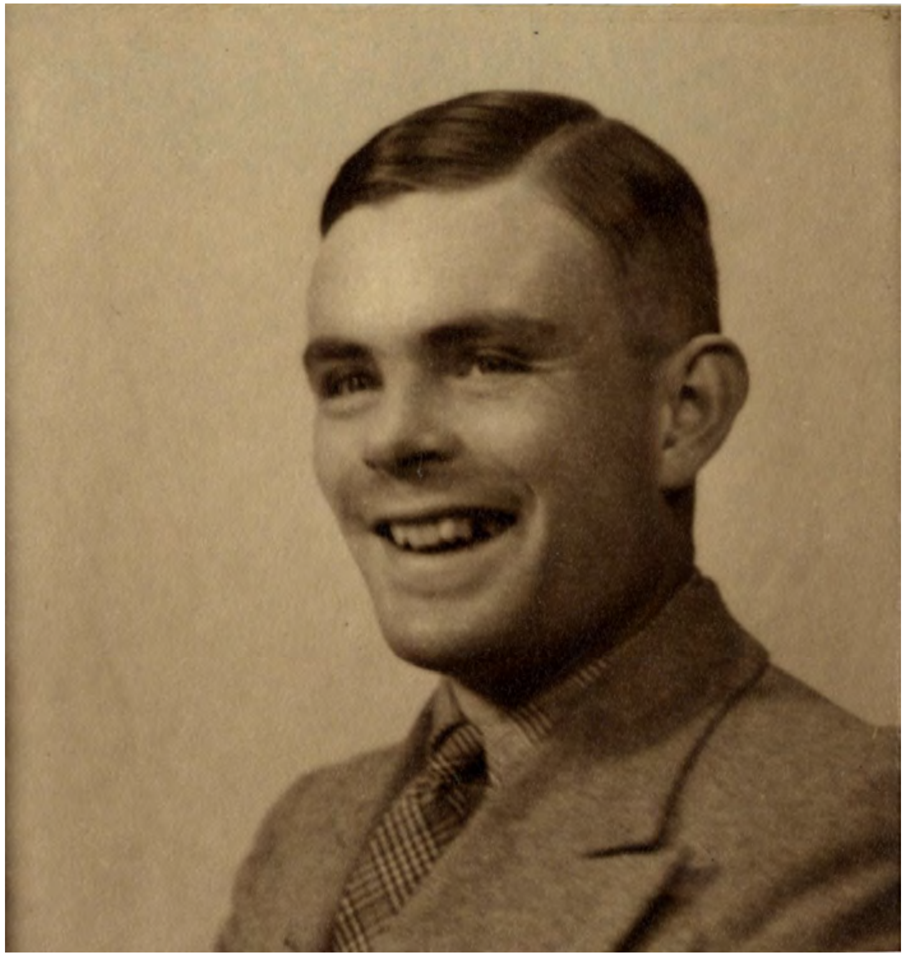
Source: Donna Haraway — Storytelling for Earthly Survival

Birthday of Donna Haraway (78 years old)

— among other things, known by the Cyborg Manifesto

Why some algorithms work a lot better than predicted?

# The Foundational Myth:
## Turing vs. Wilkinson



Source: King's College [ATM/K/7/11]

Source: Higham's web

# The Foundational Myth:

## Turing vs. Wilkinson

However, it happened that some time after my arrival, a system of 18 equations arrived in Mathematics Division and after talking around it for some time we finally decided to abandon theorizing and to solve it. A system of 18 is surprisingly formidable, even when one has had previous experience with 12, and we accordingly decided on a joint effort. The operation was manned by Fox, Goodwin, Turing, and me, and we decided on Gaussian elimination with complete pivoting. Turing was not particularly enthusiastic, partly because he was not an experienced performer on a desk machine and partly because he was convinced that it would be a failure. History repeated itself remarkably closely. Again the system was mildly ill-conditioned, the last equation had a coefficient of order $10^{-4}$ (the original coefficients being of order unity) and the residuals were again of order $10^{-10}$, that is of the size corresponding to the exact solution rounded to ten decimals. It is interesting that in connection with this example we subsequently performed one or two steps of what would now be called "iterative refinement," and this convinced us that the first solution had had almost six correct figures.

Wilkinson, 1970 Turing Lecture

# The Foundational Myth:

## Turing vs. Wilkinson

I suppose this must be regarded as a defeat for Turing since he, at that time, was a keener adherent than any of the rest of us to the pessimistic school. However, I'm sure that this experience made quite an impression on him and set him thinking afresh on the problem of rounding errors in elimination processes. About a year later he produced his famous paper "Rounding-off errors in matrix processes" [1] which together with the paper of J. von Neumann and H. Goldstine [4] did a great deal to dispel the gloom. The second round undoubtedly went to Turing!

**ROUNDING-OFF ERRORS IN MATRIX PROCESSES**

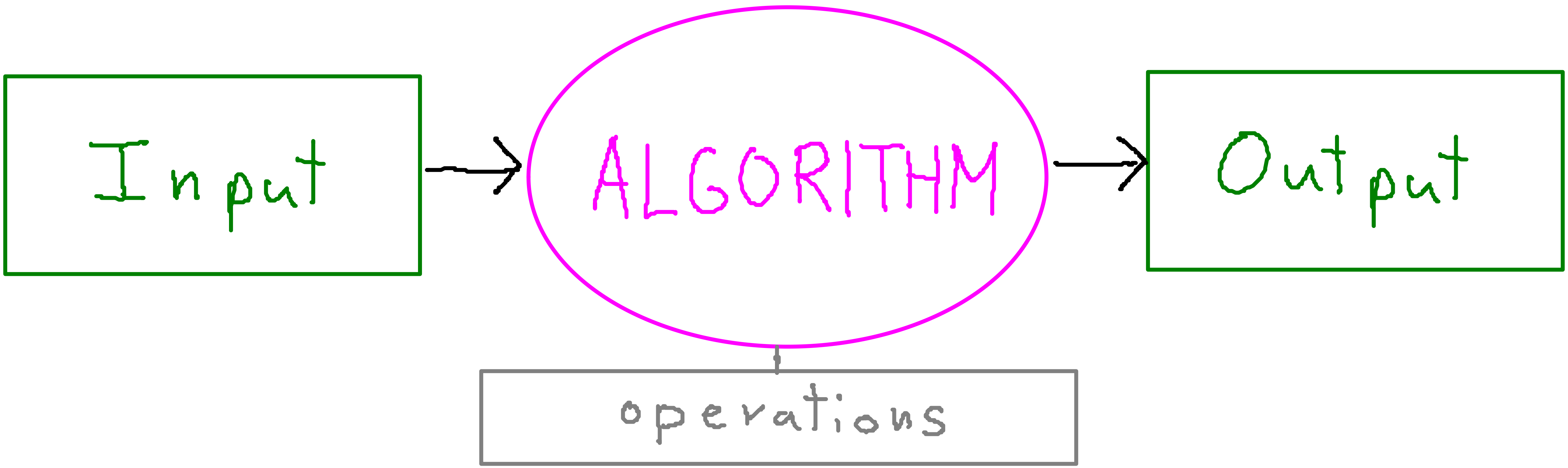
*By* A. M. TURING

(*National Physical Laboratory, Teddington, Middlesex*)

[Received 4 November 1947]

Wilkinson, 1970 Turing Lecture

# Complexity of (Traditional) Algorithms

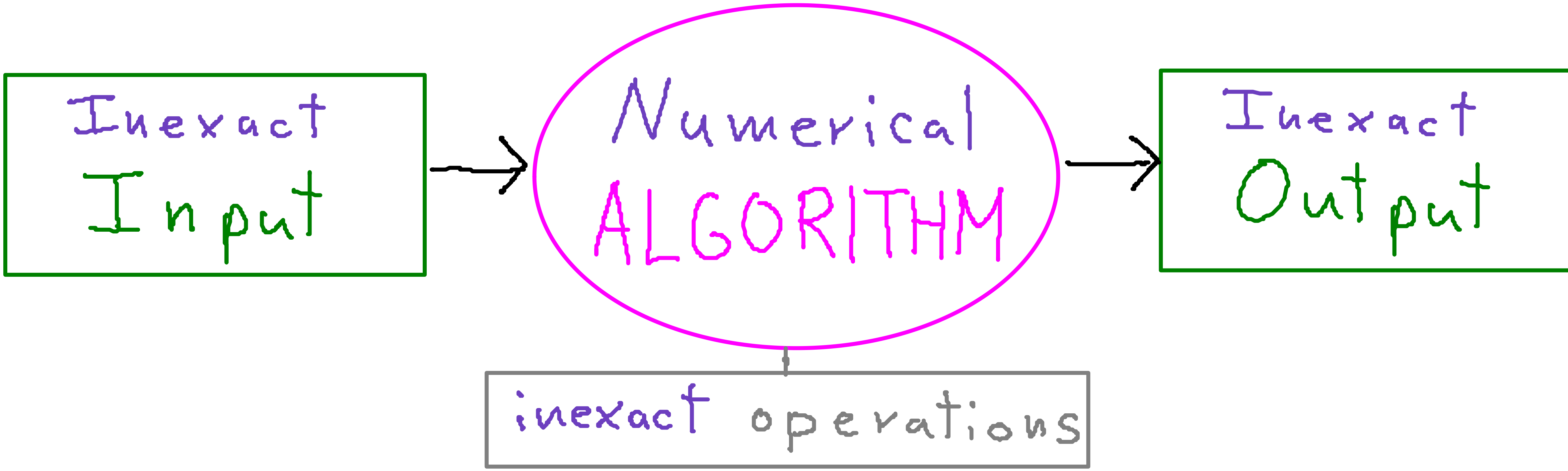



Worst-case form of complexity estimate:

$$\text{run-time}(\text{ALGORITHM}, \text{Input}) \leq g(\text{size}(\text{Input}))$$

⚠ sometimes size has several parameters
(e.g. #variables, degree...)

# Complexity of Numerical Algorithms I




⚠ usual form of complexity fails!

ALL INPUTS OF THE SAME SIZE ARE EQUAL,
BUT SOME INPUTS ARE MORE EQUAL
THAN OTHERS

# Complexity of Numerical Algorithms II

## Condition-based complexity | (Turing) (Goldstine, von Neumann)

cond(Input) : measures numerical sensitivity of Input

cond big $\Rightarrow$ small variations of Input $\longrightarrow$ big variations of Output

cond small $\Rightarrow$ 'big' variations of Input $\longrightarrow$ small variations of Output

⚠ cond is a property of the computational problem, not of the algorithm!

# Complexity of Numerical Algorithms III

Condition-based complexity II (Turing)
(Goldstine, von Neumann)

Condition-based form of complexity estimates

run-time(ALGORITHM, Input) $\leq f($size(Input), cond(Input)$)$

Can we have a complexity estimate
of a numerical algorithm only depending on size?

# Complexity of Numerical Algorithms IV

Probabilistic complexity I (Goldstine, von Neumann)
(Smale) (Demmel)

Can we have a complexity estimate of a numerical algorithm only depending on size?

Yes, if we randomize the Input

How do we randomize the Input?

We choose the probability distribution depending on the context!

Statistical complexity might have been a better name

# Complexity of (Numerical) Algorithms V

## Probabilistic complexity II (Goldstine, von Neumann) (Smale) (Demmel)

Probabilistic form of complexity estimates

$$\mathbb{P}_{\mathit{input}}\left[\text{run-time}(\text{ALGORITHM}, \mathit{input}) \geq t\right] \leq f(s,t)$$

where $\text{size}(\mathit{input}) \leq s$

... and if we are lucky

$$\mathbb{E}_{\mathit{input}}\left[\text{run-time}(\text{ALGORITHM}, \mathit{input})\right] \leq f(s)$$

# Complexity of (Numerical) Algorithms V

Smoothed complexity I (Spielman, Teng)

Smoothed form of complexity estimates

$$\sup_{\substack{\text{Input}\\ \text{size(Input)}=s}} \mathbb{P}_{\text{noise}}\left[\text{runtime}(\text{ALGORITHM}, \text{Input} + \sigma\,\text{noise}) \geq t\right] \leq f(s,t,\sigma)$$

... and if we are lucky

$$\sup_{\substack{\text{Input}\\ \text{size(Input)}=s}} \mathbb{E}_{\text{noise}}\left[\text{runtime}(\text{ALGORITHM}, \text{Input} + \sigma\,\text{noise})\right] \leq f(s,\sigma)$$

# Complexity of (Numerical) Algorithms VII

## Smoothed complexity II (Spielman, Teng)

Worst-case form of complexity estimate

$$\text{run-time}(\text{ALGORITHM}, \text{Input}) \le f(\text{size}(\text{Input}))$$

$$\uparrow \quad \sigma \to 0$$

Smoothed form of complexity estimates

$$\sup_{\substack{\text{Input} \\ \text{size}(\text{Input})=s}} \mathbb{P}_{\text{noise}}\left[\text{run-time}(\text{ALGORITHM}, \text{Input} + \sigma\,\text{noise}) \ge t\right] \le f(s,t,\sigma)$$

$$\downarrow \quad \sigma \to \infty$$

Probabilistic form of complexity estimates

$$\mathbb{P}_{\text{input}}\left[\text{run-time}(\text{ALGORITHM}, \text{input}) \ge t\right] \le f(s,t)$$

# Complexity of (Numerical) Algorithms VIII

## Success stories

### Solving Linear Equations

See any intro to numerical analysis/random matrix theory

### Linear Programming

Condition — Goffin, Renegar, Cheng, Cucker, Peña, ...

Prob./Smoothed — Bürgisser, Cucker, Lotz, Dunagan, Spielman, Teng...

### Solving Polynomial Systems

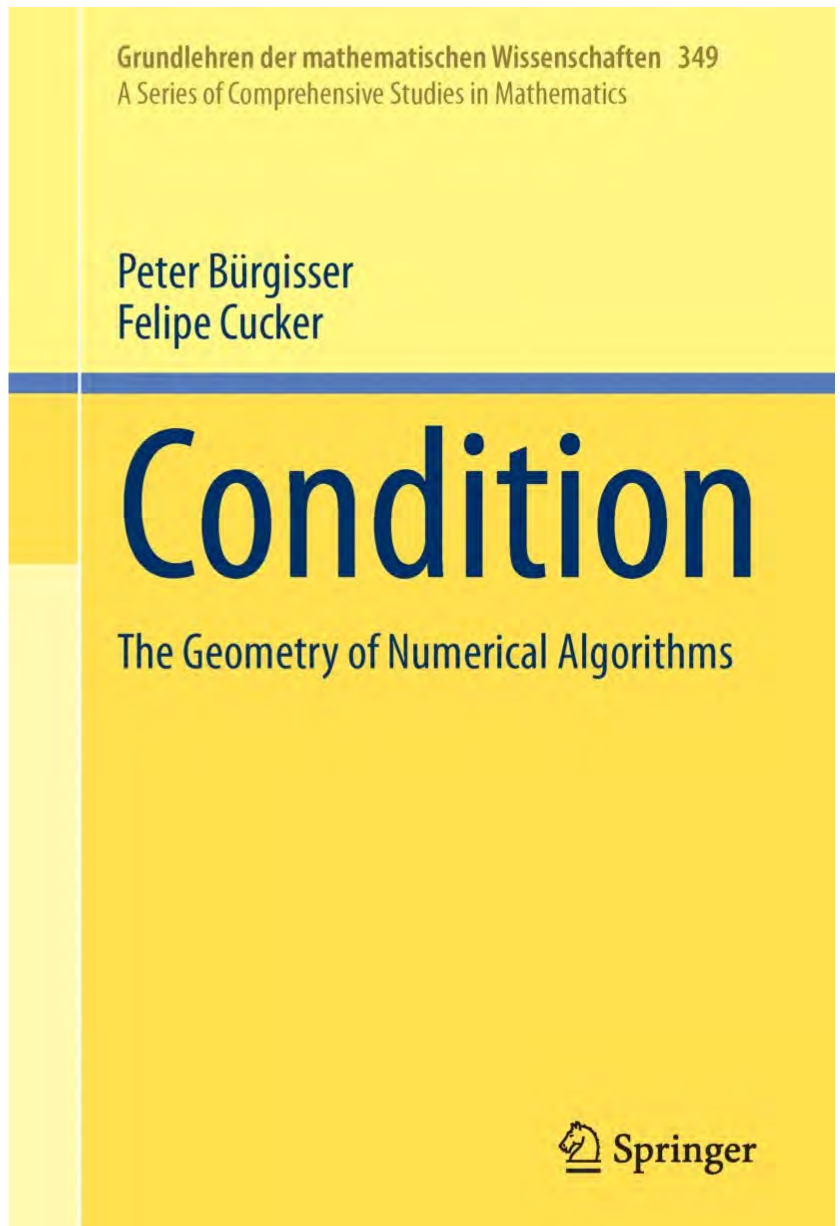
Smale's 17th Problem

Beltrán, Pardo, Bürgisser, Cucker, Lairez

...

# Complexity of (Numerical) Algorithms IX

A good introduction...



# The Framework in Action I

# the Plantinga-Vegter algorithm

Joint work with F. Cucker & A.A. Ergür
Plus extra work with E. Tsigaridas

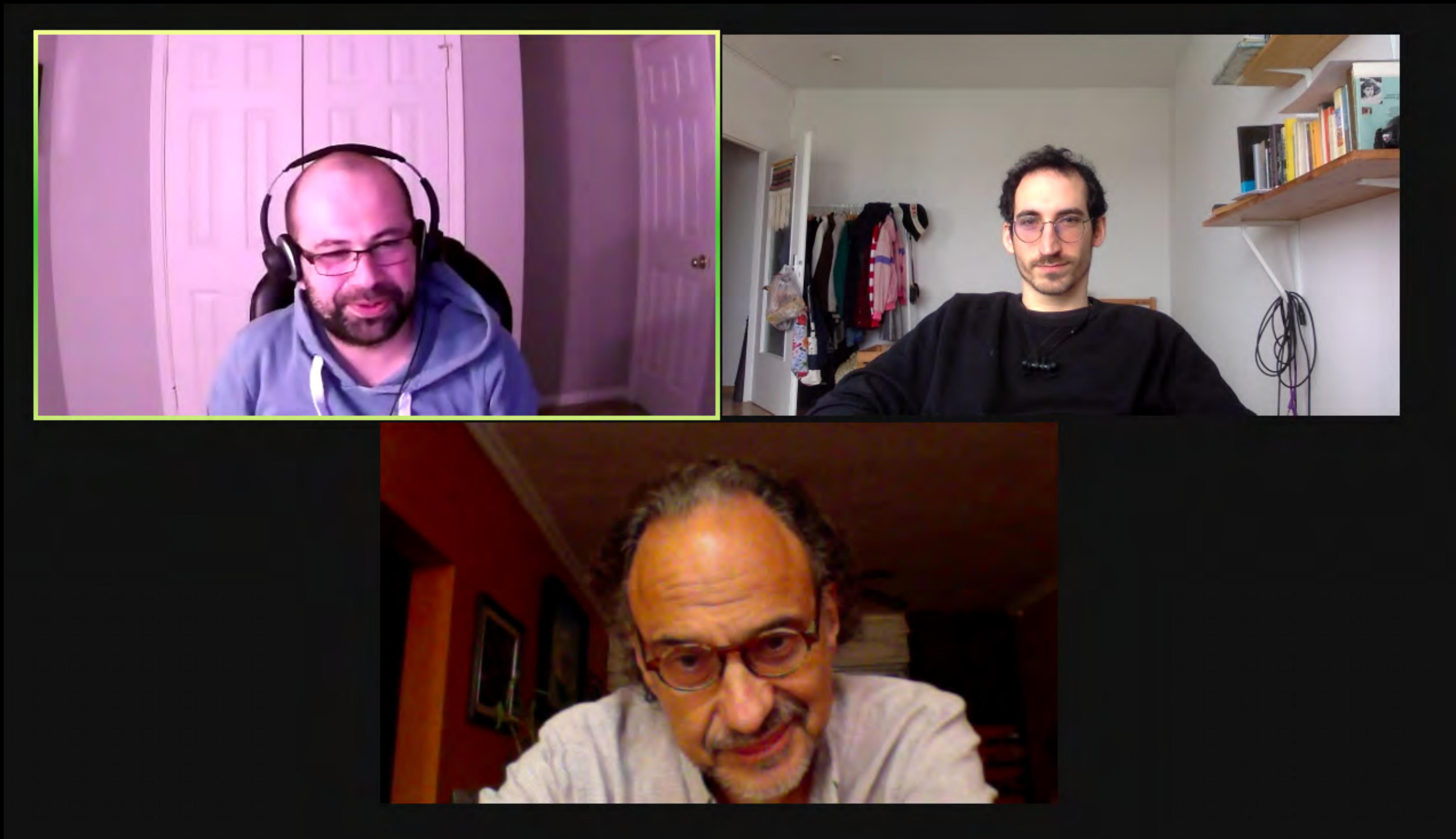


Photo while working on another project

# An algorithm for Visualizing Implicit Curves & Surfaces

Eurographics Symposium on Geometry Processing (2004)
R. Scopigno, D. Zorin, (Editors)

## Isotopic Approximation of Implicit Curves and Surfaces

Simon Plantinga and Gert Vegter

Institute for Mathematics and Computing Science
University of Groningen
simon@cs.rug.nl gert@cs.rug.nl

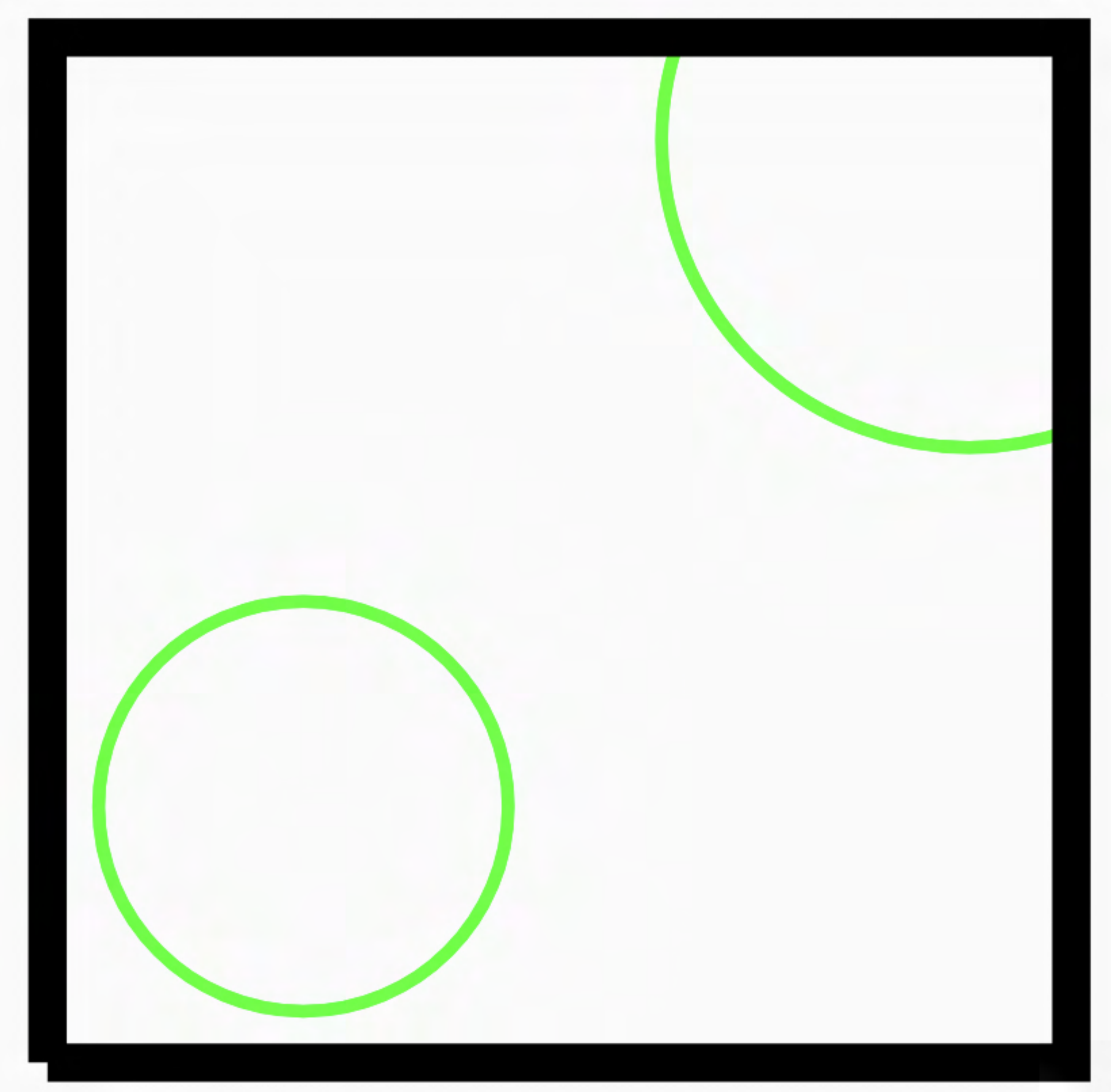
$C^1$-Function $\xrightarrow{\text{PV Algorithm}}$ PL Approximation

$g : \mathbb{R}^n \to \mathbb{R}$ of $Z(g) \cap [-a, a]^n$

Extended to hypersurfaces by Galehouse

# PV Algorithm in Action I

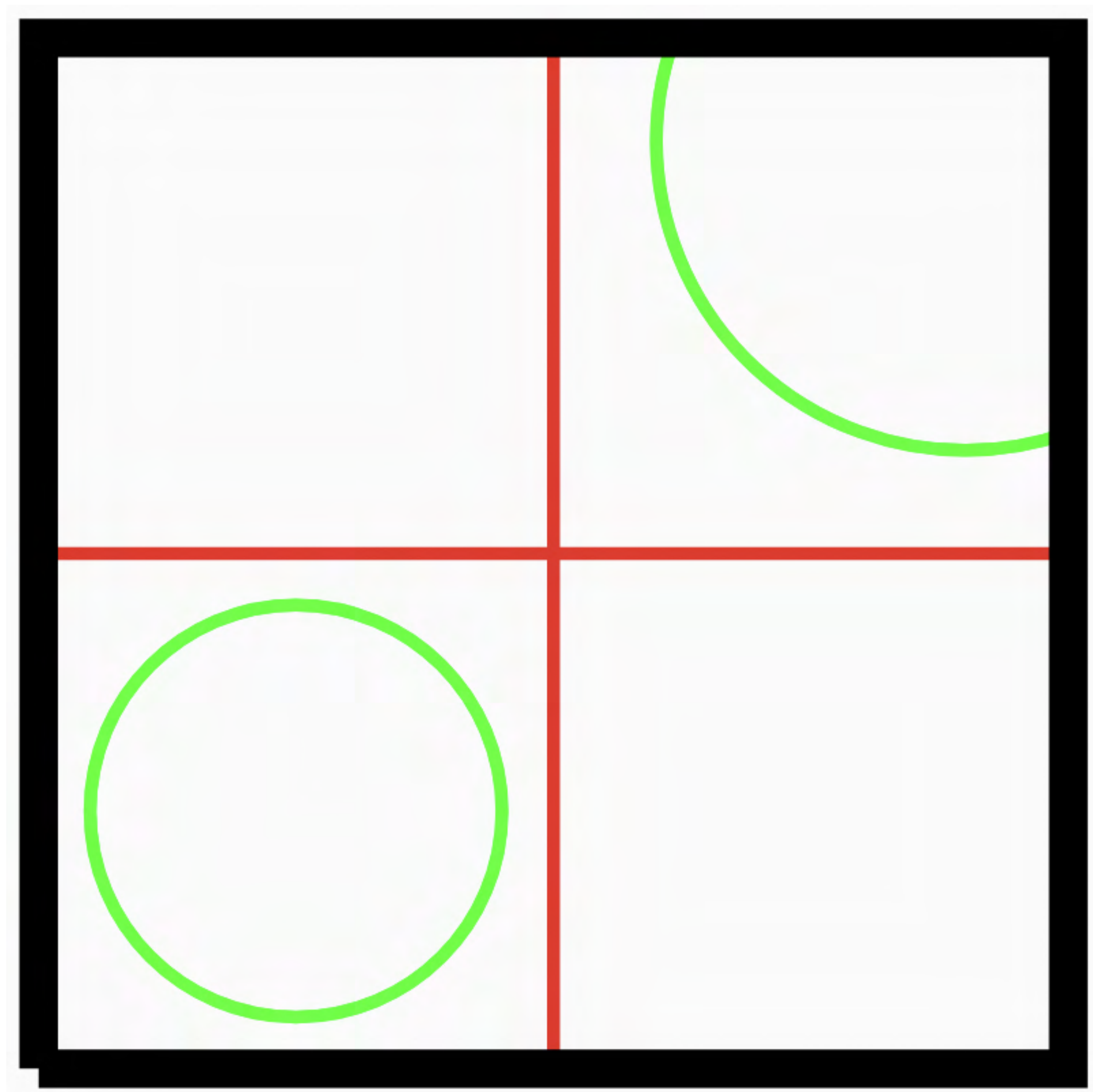
$$g = (x^2+y^2)^2 - 6(x^3+x^2y+xy^2+y^3) - 34(x^2+y^2) - 320xy + 376(x+y) + 3128$$



[-10, 10]

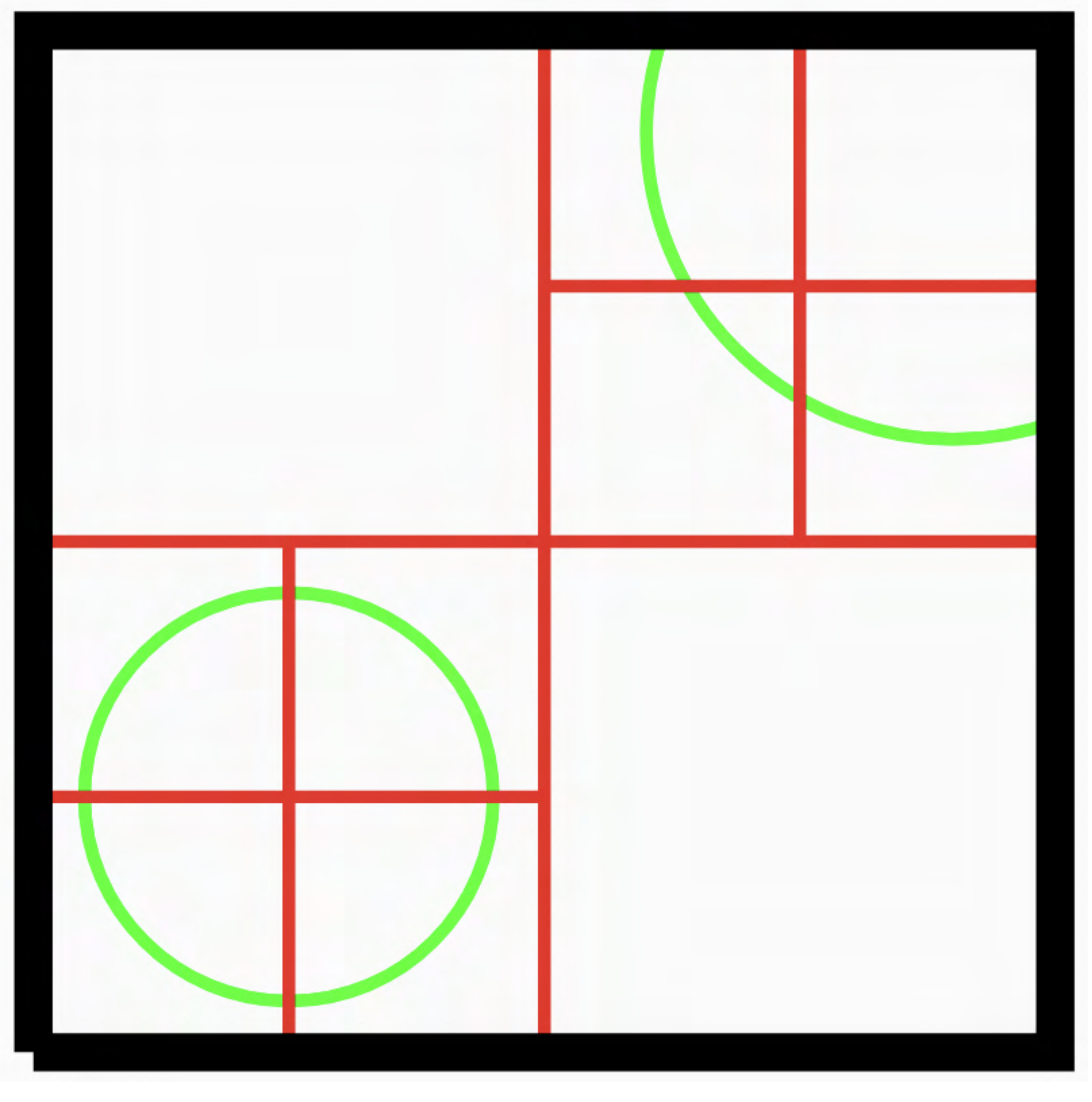
# PV Algorithm in Action I

## Subdivision Step I



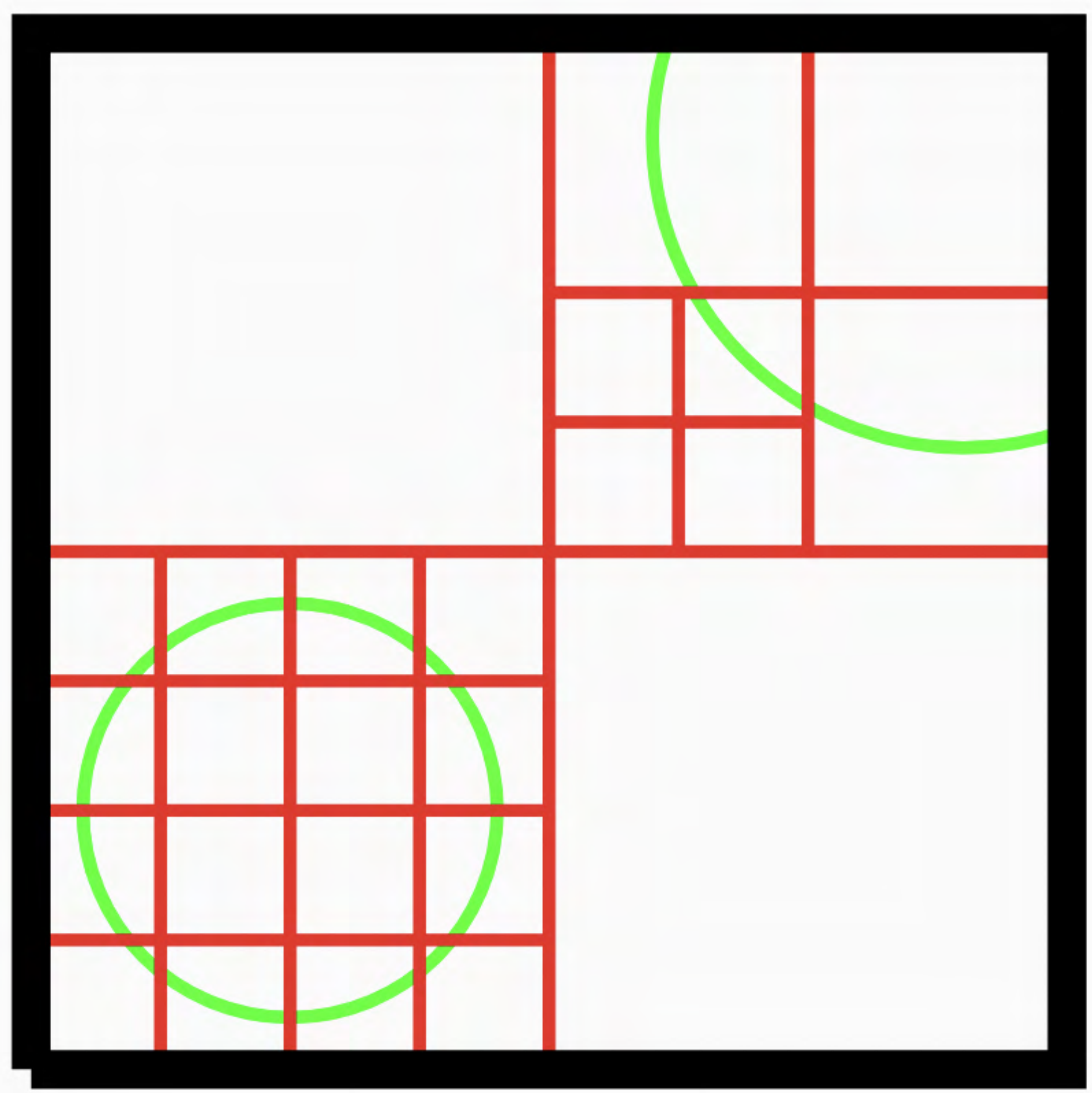
# PV Algorithm in Action I

## Subdivision Step II



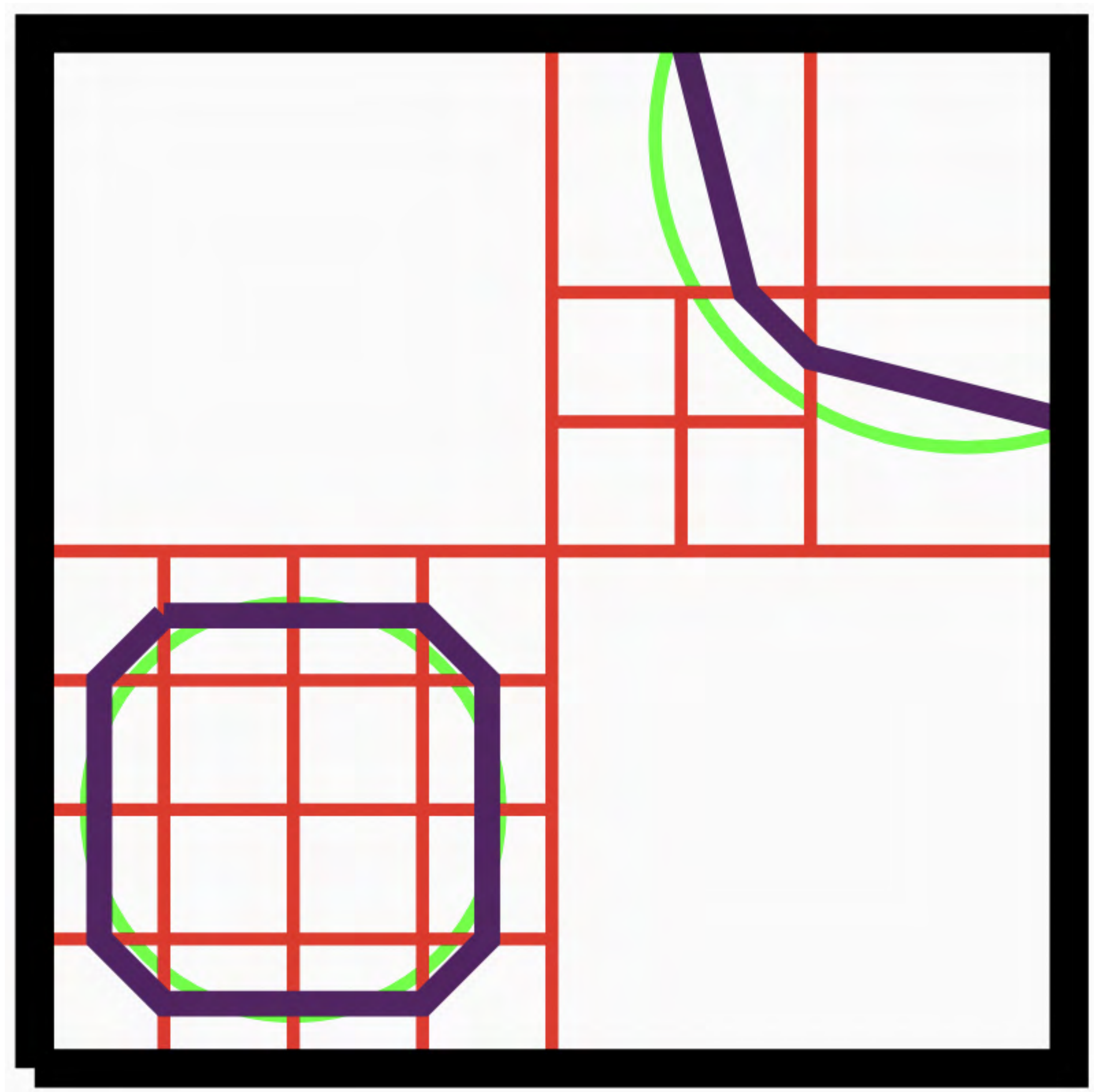
# PV Algorithm in Action I

## Subdivision Step III

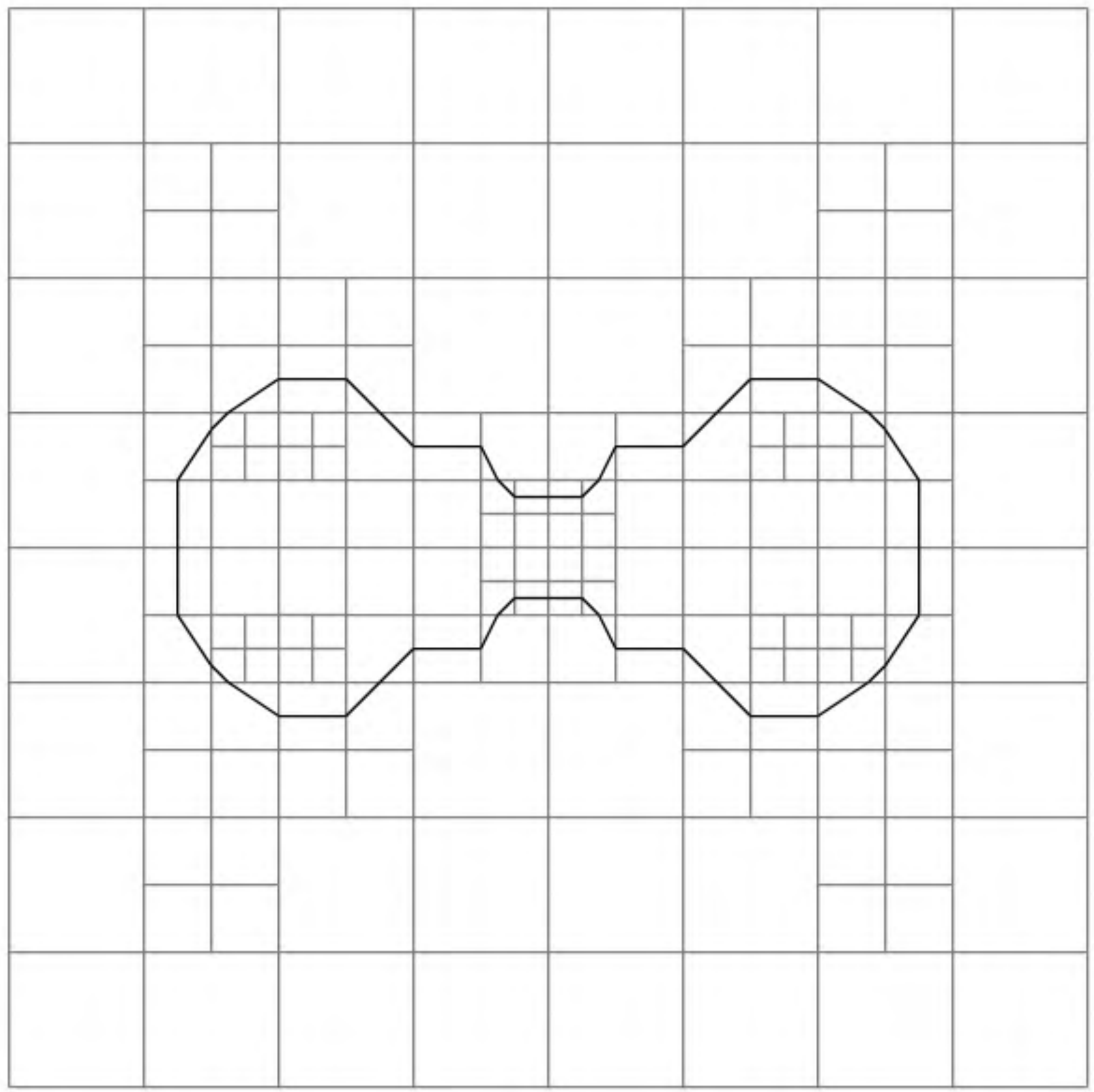


# PV Algorithm in Action I

## Postprocessing Step

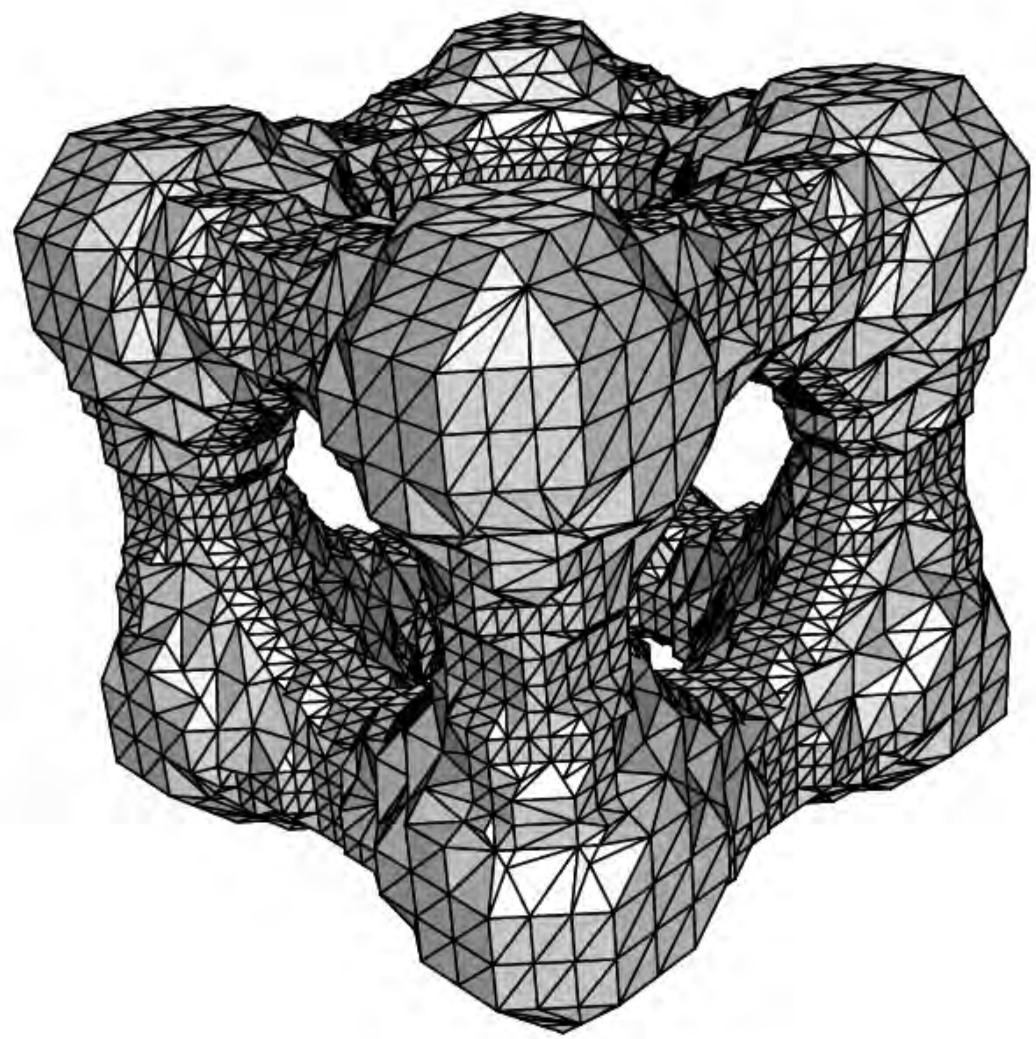


# PV Algorithm in Action III



$$g = x^2(1-x)(1+x) - y^2 + 0.01$$

# PV Algorithm in Action III



$$\delta = X^4 - 5X^2 + Y^4 - 5Y^2 + Z^4 - 5Z^2 + 10$$

PV Algorithm works in practice,

but worst-case bounds* were

too pessimistic!

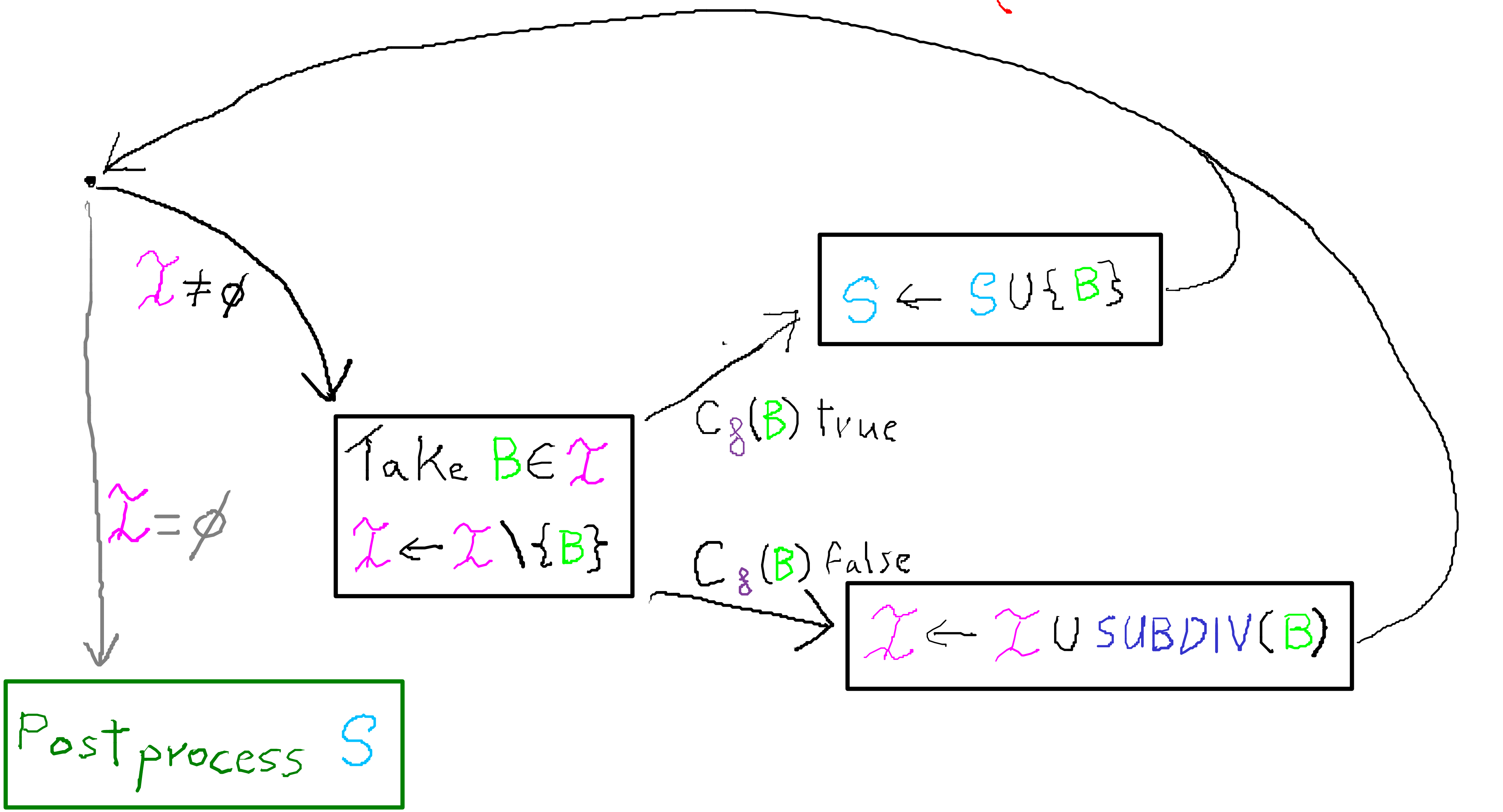
* by Burr, Gao, Tsigaridas

# Plantinga-Vegter Criterion I

$$C_g(B): \begin{cases} 0 \notin g(B) \\ \text{OR} \\ \forall x,y \in B, \quad \langle \nabla_x g, \nabla_y g \rangle \neq 0 \end{cases}$$

THM. If $S$ is a subdivision of $[-a,a]^n$ s.t. for all $B \in S$, $C_g(B)$ holds. Then we can produce a PL approx of $Z(g) \cap [-a,a]$ that is isotopically equiv.

# PLANTINGA-VEGTER ALGORITHM
## (Abstract level)




Q: How do we check $C_g(B)$?

# Interval Approximations I

Def. An <u>interval approximation</u> of

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a map

$$\square F : \square\mathbb{R}^n \longrightarrow \square\mathbb{R}^m$$

where $\square\mathbb{R}^k := \{ B \subseteq \mathbb{R}^k \mid B \text{ is a box} \}$

such that for all $B \in \square\mathbb{R}^n$,

$$F(B) \subseteq \square F(B)$$

# COMPLEXITY CONTEXT for PV

INPUT

$g \in \mathbb{R}[X_1, \dots, X_n]$

OUTPUT

PL-approximation of $\mathcal{Z}(g) \cap [-1,1]$

COMPLEXITY PARAMETERS

$d$ : degree of $g$

$n$ : number variables

$M$ : number of monomials in $g$

MEASURE OF COMPLEXITY

Size of subdivision

# Interval Approximations II

$$\|g\|_1 := \sum_\alpha |g_\alpha| \quad \text{(1-norm of } g\text{)}$$

$$\Box g(B) := g(m(B)) + d\,\|g\|_1 \left[-\frac{w(B)}{2}, \frac{w(B)}{2}\right]$$

$$\Box \|\nabla g\|_1(B) := \nabla_{m(B)} g + \sqrt{2n}\, d^2\, \|g\|_1 \left[-\frac{w(B)}{2}, \frac{w(B)}{2}\right]$$

THM. $\Box g$ is an interval approx of $g$

$\Box \|\nabla g\|_1$ is an interval approx of $\|\nabla g\|_1$

# Plantinga-Vegter Criterion II

$$C^{\square}_{g}(B): \begin{cases} 0 \notin \square g(B) \\ \text{OR} \\ 0 \notin \square \|\nabla g\|_1 (B) \end{cases}$$

THM.

$$C^{\square}_{g}(B) \Rightarrow C_{g}(B)$$

# CONDITION-BASED ESTIMATE I

## Condition number

$$C(g,x) := \frac{\|g\|_1}{\max\{|g(x)|, \|\nabla_x g\|_1\}} \in [1,\infty]$$

PROP. $C(g,x)=\infty \iff g$ singular zero at $x$

PROP. If $x \in B$ and $C(g,x)\, d\sqrt{2n}\, w(B) < 1$,

then $C^D_g(B)$ holds

# CONDITION-BASED ESTIMATE II

THM. The PV algorithm produces a subdivision with at most

$$2^{5/2 n} n^{n/2} d^n \mathbb{E}_{x \in I^n} C(g, x)^n$$

boxes on $g$

Proof relies on continuous amortization by Burr, Krahmer & Yap

# PROBABILISTIC ESTIMATE I

## UNIFORM CASE

$f := \sum_{\alpha \in A} f_\alpha X^\alpha$ $\qquad \# A =: M$

where $f_\alpha$ independent uniform in $[-1,1]$

THM. On $f$, the PV algorithm produces a subdivision with

$$2n\, 32^{n+1}\, d^{2n}\, M^{n+2}$$

boxes on average.

Similar bouds if we allow different intervals

# PROBABILISTIC ESTIMATE II
## GAUSSIAN CASE

$$f := \sum_{\alpha \in A} f_\alpha X^\alpha \qquad \# A =: M$$

where $f_\alpha$ indepent cent. Gaussians of var. 1

THM. On $f$, the PV algorithm produces a subdivision with

$$2\, n^{\frac{3}{2}} (10(n+1))^{n+1} d^{2n} M^{n+2}$$

boxes on average.

Similar bound if we allow non-centered Gaussians

# PROBABILISTIC ESTIMATE III

## GENERAL CASE: ZINTZO POLYNOMIALS

$f := \sum_{\alpha \in A} f_\alpha X^\alpha$ where $f_\alpha$ independent s.t.

(Anti-concentration)

$\forall t \in \mathbb{R}, \forall \varepsilon > 0, \ \mathbb{P}(|f_\alpha - t| < \varepsilon) < \rho_\alpha \varepsilon$

(Tail bounds)

$\forall t \in \mathbb{R}_+, \ \mathbb{P}(|f_\alpha| \geq t) \leq 2 \exp\left(-\left(t/L_\alpha\right)^p\right)$

→ Allow us to use Geometric Functional Analysis

THE MODEL IS VERY ROBUST

So
PV algorithm
is also efficient in theory!

# The Framework in Action II

# the Descartes solver

Joint work with A. A. Ergür & E. Tsigaridas

Photo while working on this project

# Real Root Isolation I: The Problem

INPUT:

$g \in \mathbb{Z}[X]$

OUTPUT:

Intervals $J_1, \dots, J_k$ s.t.

0) $J_i = (a_i, b_i)$ with $a_i, b_i \in \mathbb{Q}$

1) $Z(g) \cap \mathbb{R} \subseteq \bigcup_{i=1}^{k} J_i$

2) $\forall i, \ \# Z(g) \cap J_i = 1$

We can also handle continuous inputs!

INPUT SIZE PARAMETERS:

$d$ : degree of $g$

$\tau$ : bit-size of coefficients of $g$

MEASURE OF RUN-TIME

Bit complexity

# Real Root Isolation II: The State of the Art

Sturm Solver $\tilde{O}_B(d^4\tau^2)$

Descartes Solver $\tilde{O}_B(d^4\tau^2)$

ANewDsc $\tilde{O}_B(d^3+d^2\tau)$

(Sagraloff & Mehlhorn; 2016)

Pan's Algorithm $\tilde{O}_B(d^2\tau)$

(Pan; 2002)

Q: Can we beat the champion?

# Real Root Isolation III: What do we wish?

$$\tilde{O}_B(d\tau)$$

We wish to find real roots almost as fast as we read the polynomial!

# Descartes Solver I: Rule of Signs

Portrait by Frans Hal
Source: WikiMedia Commons

$V(g) :=$ # sign variations of $g_0, g_1, \ldots$

THM (Descartes' rule of signs)

$$\#Z(g, \mathbb{R}_>) \leq V(g)$$

Moreover,

$$V(g) \leq 1 \Rightarrow \text{Equality}$$

COR

$$\#Z(g,(a,b)) \leq V(g,(a,b)) := V\left((X+1)^d\, g\left(\frac{bX+a}{X+1}\right)\right)$$

$(0,\infty) \to (a,b)$

bijection

# Descartes Solver II:

## The Descartes' Oracle

1) Overcounting: $\#Z(g,J) \leq V(g,J)$

2) Exactness I: $V(g,J) \leq 1 \Rightarrow$ Equality

3) Exactness II:

$$\#Z(g, D(m(J), cw(J))) \leq K \Rightarrow V(g,J) \leq K$$

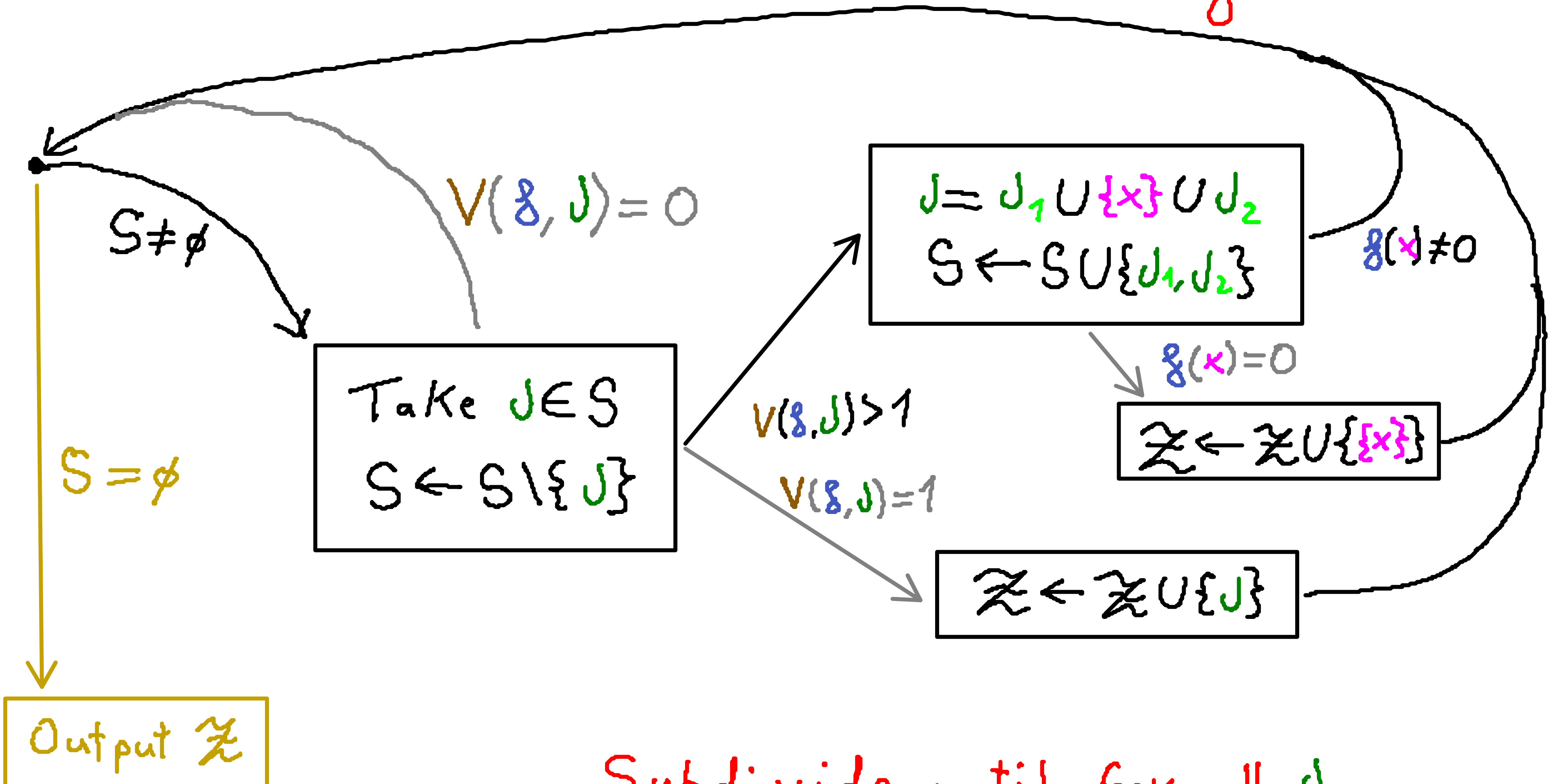
Obreshkoff's Thm: Descartes sees the complex roots around!

4) Subadditivity:

$$\dot{\bigcup} J_i \subseteq J \Rightarrow \sum V(g, J_i) \leq V(g,J)$$

# Descartes Solver III:

## The Algorithm

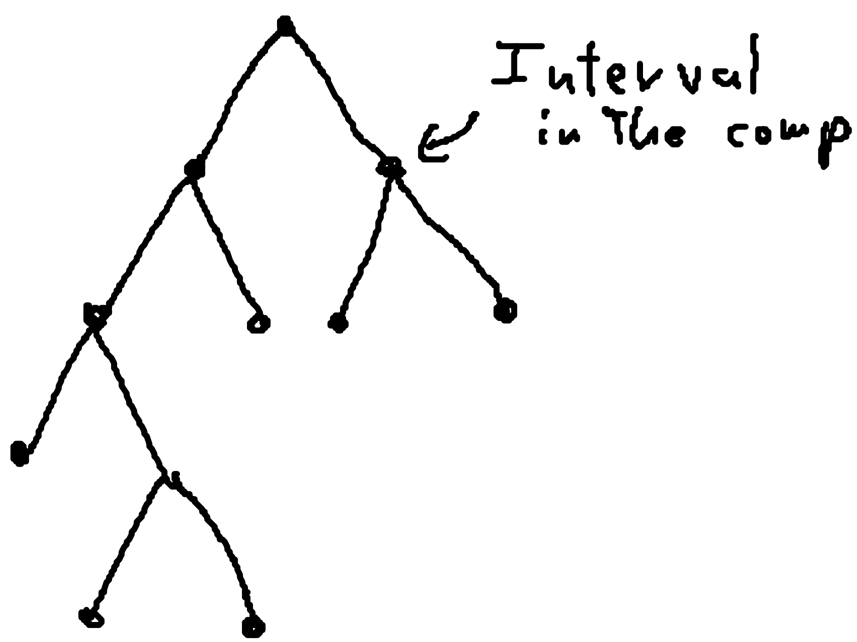



Subdivide until for all $J$,

$V(f, J) \leq 1$!

# Descartes Solver IV: Descartes' tree

$\mathcal{T}(g, I)$




size of $\mathcal{T}(g, I)$

$\updownarrow$

run-time of $\text{DESCARTES}(g, I)$

We only need to control the size of subdiv. tree!

# Real Root Isolation IV:
## Are we being pessimistic?

DESCARTES SOLVER
seems to behave faster in practice!

¿Can we explain this?

# Real Root Isolation V:
# Beyond pessimism

Yes, bounding

$$\mathbb{E}\left\{\mathrm{cost}(\mathrm{SOLVER}, f)^{\ell} \,\middle|\, \text{bit-size}(f) \le \tau,\ \deg(f) \le d\right\}$$

What's a 'good' random model for $f$?

Many choices of randomness 😱

# Beyond pessimism I: Uniform Random Bit Polynomials & a Simple Main Theorem

$$F = \sum_{k=0}^{d} f_k X^k$$

s.t. $f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$ independent

SIMPLE MAIN THM

$$\mathbb{E}\, \mathrm{cost}(\text{DESCARTES}, F) = \widetilde{\mathcal{O}}_B(d^2 + d\tau)$$

On average, DESCARTES is almost optimal!

# Beyond pessimism II: Random Bit Polynomials

$$f=\sum_{k=0}^{d} f_k X^k \in \mathbb{Z}[X]$$

s.t. $f_k$ independent

bit-size of $f$:

$$\tau(f):=\min\{\tau \mid \forall k, \mathbb{P}(|f_k|\leq 2^{\tau})=1\}$$

weight of $f$:

No middle indexes!

$$w(f):=\max\{\mathbb{P}(f_k=c) \mid c\in\mathbb{R}, k\in\{0,1,d-1,d\}\}$$

uniformity of $f$: $u(f):=\ln\left(w(f)\left(1+2^{\tau(f)+1}\right)\right)$

# Beyond pessimism III: Main Theorem

## Main Thm

$$\mathbb{E}\,\mathrm{cost}(\text{DESCARTES}, f) = \widetilde{O}_B\left(d^2 + d\tau\right)\left(1 + u(f)\right)^4$$

Note: $f$ uniform $\Rightarrow u(f) = 0$

Claim: For many cases, $u(f) = O(1)$

IF $\tau = \Omega(d)$, almost like reading!

On average, DESCARTES is almost optimal!

# Beyond pessimism IV: Examples of Random Bit Polynomials I

- Support control $\{0,1,d-1,d\} \subseteq A$

  $F = \sum_{k \in A} f_k X^k$ with $f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$

  ... then $u(F) = 0$

- Sign control $\sigma \in \{-1,+1\}^{\{0,\dots,d\}}$

  $F = \sum_{k=1}^{d} f_k X^k$ with $f_k \sim \mathcal{U}(\sigma_k([1,2^\tau] \cap \mathbb{N}))$

  ... then $u(F) \leq \ln 3$

# Beyond pessimism V: Examples of Random Bit Polynomials II

- Exact bitsize

$$F = \sum_{k=1}^{d} f_k X^k \quad \text{with} \quad f_k \sim \mathcal{U}(\{n \in \mathbb{Z} \mid \lfloor \log n \rfloor = \tau\})$$

... then $u(F) \leq \ln 3$

+ their combinations

Our random model is flexible!

# Beyond pessimism $\boxed{V}$: Smoothed case included!

$$F = \sum_{k=1}^{d} f_k X^k$$ random bit polynomial

$$g = \sum_{k=1}^{d} g_k X^k$$ fix polynomial of entries of size $\tau$

$$\sigma \in \mathbb{Z} \setminus \{0\}$$

Then:

$$F_\sigma = g + \sigma f$$ random bit polynomial

$$\& \quad u(F_\sigma) \leq 1 + u(F) + \max\{\tau - \tau(F), \tau(\sigma)\}$$

# The Ingredients of the Analysis I: Condition Numbers

$$C(g) := \max_{x \in [-1,1]} \frac{\sum_{k=0}^{d} |g_k|}{\max\{|g(x)|, |g'(x)|/d\}}$$

$C(g) = \infty \Leftrightarrow g$ has a singular root in $[-1,1]$

Upper bounds on $C(g)$

→ Lower bounds for root separation of $g$

→ Upper bounds for depth of DESCARTES' tree

# The Ingredients of the Analysis II: Bounds for Number of Complex Roots

We only care about nearby roots!

Upper bounds for
# complex roots of $g$ around $[-1,1]$

$\longrightarrow$ Upper bounds for width of DESCARTES' tree

Complex analysis!
Tichmarsh thm

# The Ingredients of the Analysis III: Probabilistic Toolbox

Ball's smoothing:

$x \in \mathbb{Z}^N$ discrete random variable

$y \in \mathbb{R}^N$ s.t. $y_i \sim \mathcal{U}((-\frac{1}{2}, \frac{1}{2}))$ i.i.d.

Then: $x + y$ continuous random var.

We can use our old cont. toolbox!

⚠ I am omitting a lot of technical details.

Summing up:

Descartes

is almost optimal on average!

Eskerrik Asko

zure arretagatik!

I.e. Thank You for your attention!