Probabilistic Bounds on Best Rank-One Approximation Ratio

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Tensors

A *tensor* is a multi-indexed list of numbers, i.e., a map

$$\{1,\dots,n_1\}\times\cdots\times\{1,\dots,n_d\}\ni(j_1,\dots,j_d)\mapsto t_{j_1,\dots,j_d}$$

We denote the space of these tensors by $\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$.

So...
A *vector* is a list of numbers
A *matrix* is a table of numbers
A *tensor* is a multidimensional box of numbers
—a *1-tensor* is a vecor and a *2-tensor* a matrix

Rank-One Tensors

A *rank-one tensor* is a tensor $\lambda\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d$ of the form

$$\lambda\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d := \left(\lambda x^1_{j_1}\cdots x^d_{j_d}\right)$$

where λ is an scalar and the $\mathbf{x}^i$ are vectors.

Observation. If $\lambda\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d$ is a real tensor, then we can assume without loss of generality that $\lambda,\mathbf{x}^1,\dots,\mathbf{x}^d$ are real.

So...
Every 1-tensor (vector) is rank-one
A rank-one 2-tensor is just a rank-one matrix

Frobenius norm for tensors

Given a tensor $\mathrm{T}=(t_{i_1,\dots,i_d})$, its *Frobenius norm* is

$$\|\mathrm{T}\| := \sqrt{\sum_{j_1,\dots,j_d}|t_{j_1,\dots,j_d}|^2}.$$

This norm induces a Hermitian inner product that we denote by $\langle\ ,\ \rangle$.

So...
In the case of 2-tensors (matrices), this agrees with the usual definition

Best rank-one approximation

Given a tensor $\mathrm{T}=(t_{i_1,\dots,i_d})\in\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$, a *best rank-one approximation* of T is a rank-one tensor $\alpha\mathbf{z}^1\otimes\cdots\otimes\mathbf{z}^d\in\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$ such that for every rank-one tensor $\lambda\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d\in\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$,

$$\|\mathrm{T}-\alpha\mathbf{z}^1\otimes\cdots\otimes\mathbf{z}^d\|\le\|\mathrm{T}-\lambda\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d\|.$$

Motivating question.
How bad can a rank-one approximation be?

Note...
When working with real tensors,
we limit ourselves to real best rank-one approximations

Best Rank-One Approximation Ratio

Qi & Lu (2017) showed that for a tensor $\mathrm{T}\in\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$, any of its best rank-one approximations $\alpha\mathbf{z}^1\otimes\cdots\otimes\mathbf{z}^d\in\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$ satisfies

$$\frac{\|\mathrm{T}-\alpha\mathbf{z}^1\otimes\cdots\otimes\mathbf{z}^d\|^2}{\|\mathrm{T}\|^2}=1-\left(\max_{\mathbf{x}^i\in\mathbb{K}^{n_i}}\left\{\frac{|\langle\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d,\mathrm{T}\rangle|}{\|\mathbf{x}^1\|\cdots\|\mathbf{x}^d\|\|\mathrm{T}\|}\right\}\right)^2.$$

Moreover, the maximum on the right-hand side is achieved at $\mathbf{z}^1\otimes\cdots\otimes\mathbf{z}^d$.

Best Rank-One Approximation Ratio for X

Given a linear subspace $\mathrm{X}\subseteq\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d}$, the *rank-one approximation ratio* for X is

$$\mathcal{A}(\mathrm{X}) := \min_{\mathrm{T}\in\mathrm{X}}\max_{\mathbf{x}^i\in\mathbb{K}^{n_i}}\frac{|\langle\mathbf{x}^1\otimes\cdots\otimes\mathbf{x}^d,\mathrm{T}\rangle|}{\|\mathbf{x}^1\|\cdots\|\mathbf{x}^d\|\|\mathrm{T}\|}\in(0,1].$$

What does $\mathcal{A}(\mathrm{X})$ measure? The quality of the worst-approximating best-rank approximation of tensors in X

Bounds for $\mathcal{A}(\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d})$ ($\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$)

The result was known, we provided an explicit upper bound...

Theorem

$$\frac{1}{\sqrt{\min_j\prod_{i\neq j}n_i}}\le\mathcal{A}(\mathbb{K}^{n_1}\otimes\cdots\otimes\mathbb{K}^{n_d})\le\frac{10\sqrt{d\ln d}}{\sqrt{\min_j\prod_{i\neq j}n_i}}$$

Note the bound does not care if $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$!

Proof techniques...

Geometric Functional Analysis, Integral Identities & Probability!

All the details available at...
arXiv:2201.02191

What about symmetric tensors?

A *symmetric tensor* $\mathrm{T}=(t_{j_1,\dots,j_d})\in(\mathbb{K}^n)^{\otimes d}:=\mathbb{K}^n\otimes\cdots\otimes\mathbb{K}^n$ is a tensor such that for every permutation $\sigma\in\Sigma_d$ and all $(j_1,\dots,j_d)$,

$$t_{j_1,\dots,j_d}=t_{j_{\sigma(1)},\dots,j_{\sigma(d)}}.$$

We denote by $\mathrm{Sym}^d(\mathbb{K}^n)\subseteq(\mathbb{K}^n)^{\otimes d}$ the subspace of symmetric tensors.

Bounds for $\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))$ & $\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))$

Complex Theorem (K. & T.-C., '22 +)

For any $d\ge3$ and $n\ge2$,

$$\max\left\{\binom{d+n-1}{d}^{-\frac12},\frac{1}{n^{\frac{d-1}{2}}}\right\}\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))\le10\sqrt{n\ln d}\binom{d+n-1}{d}^{-\frac12}.$$

In particular,

$$\frac{1}{n^{\frac{d-1}{2}}}\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))\le6\left(1+\frac{1}{\ln d}\right)\sqrt{d!\ln d}\frac{1}{n^{\frac{d-1}{2}}},$$

and, for $d\ge n^2/4$,

$$\sqrt{\frac{(n-1)!}{d^{n-1}}}\left(1-\frac{n^2}{4d}\right)\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))\le10\sqrt{\frac{n!\ln d}{d^{n-1}}}.$$

Real Theorem (K. & T.-C., '22 +)

For any $d\ge3$ and $n\ge2$,

$$\max\left\{\frac{1}{2^{\frac{d}{2}}}\binom{d+n-1}{d}^{-\frac12},\frac{1}{n^{\frac{d-1}{2}}}\right\}\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))\le\frac{6\sqrt{n\ln d}}{2^{\frac{d}{2}}}\binom{d+\frac{n}{2}-1}{d}^{-\frac12}.$$

In particular,

$$\frac{1}{n^{\frac{d-1}{2}}}\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))\le6\left(1+\frac{1}{\ln d}\right)\sqrt{d!\ln d}\frac{1}{n^{\frac{d-1}{2}}},$$

and, for $d\ge n^2/4$,

$$\sqrt{\frac{(n-1)!}{2^d d^{n-1}}}\left(1-\frac{n^2}{4d}\right)\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))\le9\sqrt{\frac{\left(\frac{n}{2}\right)!\ln d}{2^d d^{\frac{n}{2}-1}}}\left(1+\frac{1}{4d}\right).$$

Corollary (K. & T.-C., '22 +)

For a fixed $d\ge3$, there is a constant $\mathrm{C}_d>0$ (depending on d) such that

$$\mathcal{A}((\mathbb{K}^n)^{\otimes d})\le\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n)),\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))\le\mathrm{C}_d\mathcal{A}((\mathbb{K}^n)^{\otimes d}).$$

Corollary (K. & T.-C., '22 +)

For a fixed $n\ge3$,

$$\lim_{d\to\infty}\frac{\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))}{\mathcal{A}(\mathrm{Sym}^d(\mathbb{C}^n))}=\lim_{d\to\infty}\frac{\mathcal{A}((\mathbb{K}^n)^{\otimes d})}{\mathcal{A}(\mathrm{Sym}^d(\mathbb{R}^n))}=0.$$

Also for partially symmetric tensors!