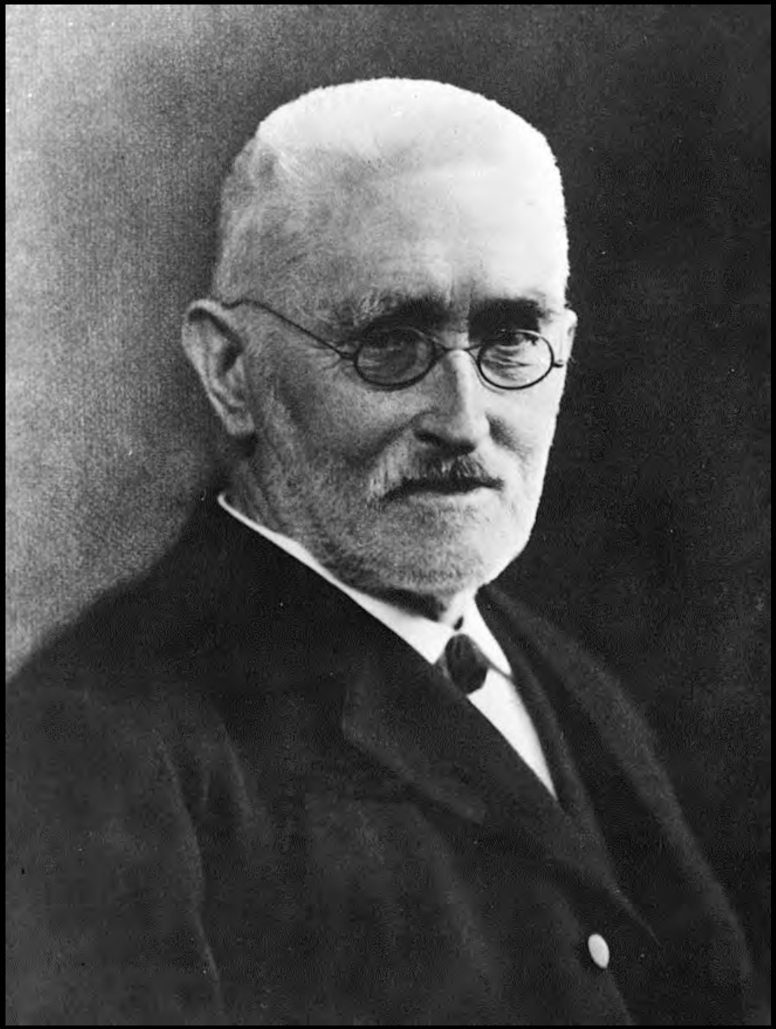
Fact of the Day



Today, October 6, Richard Dedekind would have turned 191 years old

8TH IBERIAN MATHEMATICAL MEETING
Sevilla, Spain

Why does the DESCARTES Solver work?

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Photo while working on this project

Real Root Isolation I: The Problem

INPUT:

$f \in \mathbb{Z}[X]$

OUTPUT:

Intervals $J_1, \dots, J_k$ s.t.

0) $J_i = (a_i, b_i)$ with $a_i, b_i \in \mathbb{Q}$

1) $Z(f) \cap \mathbb{R} \subseteq \bigcup_{i=1}^{k} J_i$

2) $\forall i, \# Z(f) \cap J_i = 1$

INPUT SIZE PARAMETERS:

d : degree of f

τ : bit-size of coefficients of f

MEASURE OF RUN-TIME

Bit complexity

Real Root Isolation II: The State of the Art

Sturm Solver	$\tilde{O}_B(d^4 \tau^2)$
Descartes Solver	$\tilde{O}_B(d^4 \tau^2)$
ANewDsc (Sagraloff & Mehlhorn; 2016)	$\tilde{O}_B(d^3 + d^2 \tau)$
Pan's Algorithm (Pan; 2002)	$\tilde{O}_B(d^2 \tau)$

Q: Can we beat the champion?

Real Root Isolation III:

What do we wish?

$$\tilde{O}_B(d\tau)$$

We wish to find real roots almost as fast as we read the polynomial!

Descartes Solver I: Rule of Signs

Portrait by Frans Hal
Source: WikiMedia Commons

$V(g) :=$ # sign variations of $g_0, g_1, \dots$

THM (Descartes' rule of signs)

$$\#Z(g, \mathbb{R}_>) \le V(g)$$

Moreover,

$$V(g) \le 1 \Rightarrow \text{Equality}$$

COR

$$\#Z(g,(a,b)) \le V(g,(a,b)) := V\left((X+1)^d \, g\left(\frac{bX+a}{X+1}\right)\right)$$

$(0,\infty) \longrightarrow (a,b)$

bijection

Descartes Solver II: The Descartes' Oracle

1) Overcounting: $\#Z(g,J) \leq V(g,J)$

2) Exactness I: $V(g,J) \leq 1 \Rightarrow$ Equality

3) Exactness II:

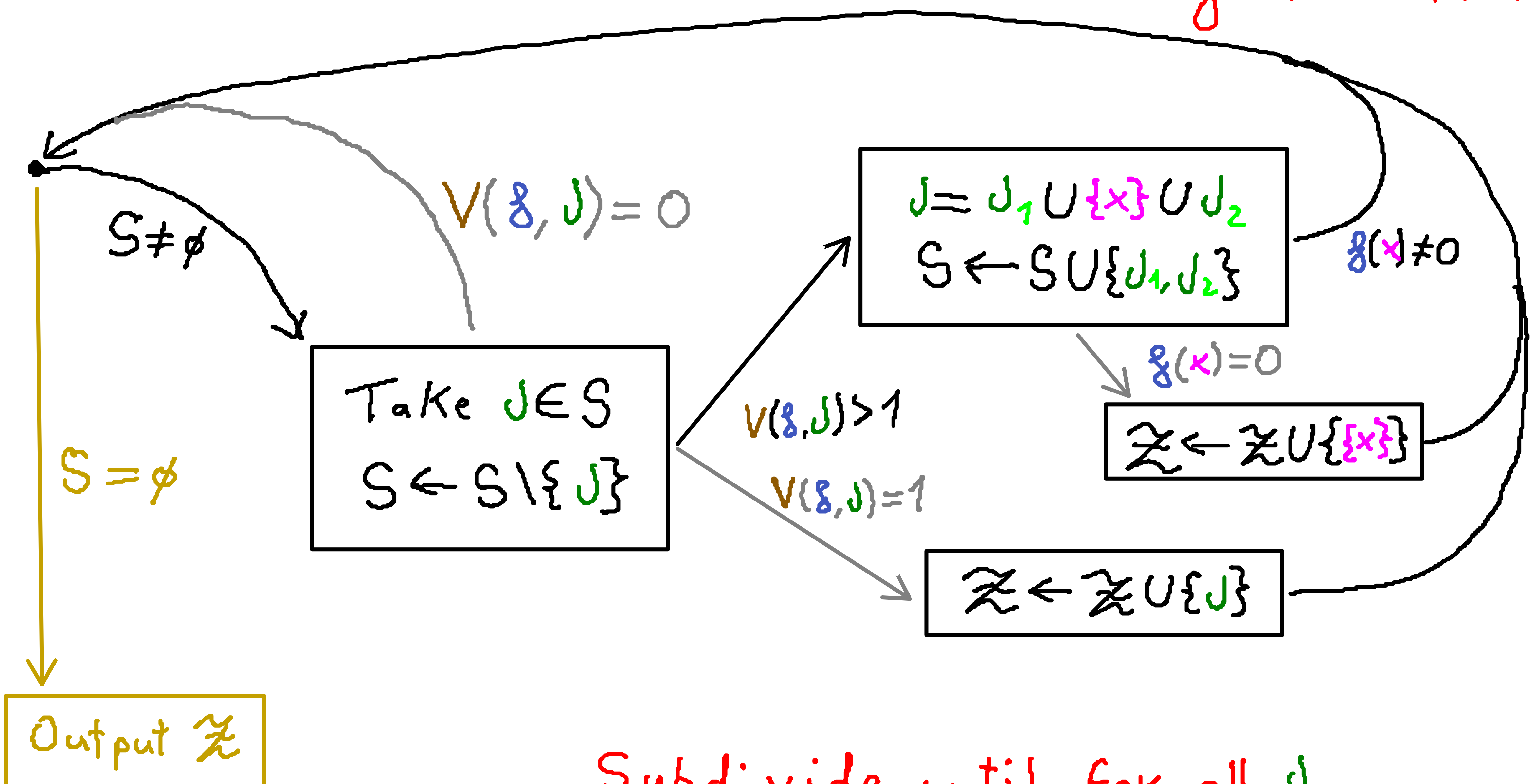
$$\#Z(g, D(m(J), c\,w(J))) \leq K \Rightarrow V(g,J) \leq K$$

Obreshkoff's Thm: Descartes sees the complex roots around!

4) Subadditivity:

$$\dot{\bigcup} J_i \subseteq J \Rightarrow \sum V(g, J_i) \leq V(g, J)$$

Descartes Solver III: The Algorithm



Subdivide until for all J,

$V(f, J) \leq 1$!

Real Root Isolation IV:

Are we being pessimistic?

Worst-case complexity:

$$\max\{\mathrm{cost}(\text{SOLVER}, f) \mid \text{bit-size}(f) \leq \tau, \deg(f) \leq d\}$$

Pessimistic in practice

Descartes Solver seems to behave faster in practice!

¿Can we explain this?

Real Root Isolation V: Beyond pessimism

Worst-case complexity:

$$\max\{\mathrm{cost}(\mathrm{SOLVER}, g) \mid \text{bit-size}(g) \leq \tau, \deg(g) \leq d\}$$

(Goldstine & von Neumann, 1951)
(Demmel, 1988) (Smale; 1985, 1997)

(Roughgarden, 2021)

Probabilistic complexity

$$\mathbb{E}\{\mathrm{cost}(\mathrm{SOLVER}, \mathfrak{F})^{\ell} \mid \text{bit-size}(\mathfrak{F}) \leq \tau, \deg(\mathfrak{F}) \leq d\}$$

What's a 'good' random model for $\mathfrak{F}$?

Many choices of randomness 😱

Beyond pessimism I: Uniform Random Bit Polynomials & a Simple Main Theorem

$$f = \sum_{k=0}^{d} f_k X^k$$

s.t. $f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$ independent

SIMPLE MAIN THM (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\,\mathrm{cost}(\mathrm{DESCARTES}, f) = \widetilde{\mathcal{O}}_B(d^2 + d\tau)$$

On average, DESCARTES is almost optimal!

Beyond pessimism II: Random Bit Polynomials

$$f = \sum_{k=0}^{d} f_k X^k \in \mathbb{Z}[X]$$

s.t. f_k independent

bit-size of f:

$$\tau(f) := \min\{\tau \mid \forall k, \mathbb{P}(|f_k| \leq 2^{\tau}) = 1\}$$

weight of f:

No middle indexes!

$$w(f) := \max\{\mathbb{P}(f_k = c) \mid c \in \mathbb{R}, k \in \{0, 1, d-1, d\}\}$$

uniformity of f: $u(f) := \ln\left(w(f)\left(1 + 2^{\tau(f)+1}\right)\right)$

Beyond pessimism III:

Main Theorem

Main Thm (Ergür, T-C, Tsigaridas)

$$\mathbb{E}\,\mathrm{cost}(\mathrm{DESCARTES}, f) = \widetilde{\mathcal{O}}_B\left(d^2 + d\tau\right)\left(1+u(f)\right)^4$$

Note: f uniform $\Rightarrow u(f) = 0$

Claim: For many cases, $u(f) = \mathcal{O}(1)$

IF $\tau = \Omega(d)$, almost like reading!

On average, DESCARTES is almost optimal!

Beyond pessimism IV: Examples of Random Bit Polynomials I

- Support control $\{0, 1, d-1, d\} \subseteq A$

$$f = \sum_{k \in A} f_k X^k \text{ with } f_k \sim \mathcal{U}([-2^\tau, 2^\tau] \cap \mathbb{Z})$$

... then $u(f) = 0$

- Sign control $\sigma \in \{-1, +1\}^{\{0, \ldots, d\}}$

$$f = \sum_{k=1}^{d} f_k X^k \text{ with } f_k \sim \mathcal{U}(\sigma_k([1, 2^\tau] \cap \mathbb{N}))$$

... then $u(f) \leq \ln 3$

Beyond pessimism V:
Examples of Random Bit Polynomials II

- Exact bitsize

$$F = \sum_{k=1}^{d} f_k X^k \quad \text{with} \quad f_k \sim \mathcal{U}(\{n \in \mathbb{Z} \mid \lfloor \log n \rfloor = \tau\})$$

... then $u(F) \leq \ln 3$

+ their combinations

Our random model is flexible!

Beyond pessimism V:
Smoothed case included!

$F = \sum_{k=1}^{d} f_k X^k$ random bit polynomial

$g = \sum_{k=1}^{d} g_k X^k$ fix polynomial of entries of size τ

$\sigma \in \mathbb{Z} \setminus \{0\}$

Then:

$F_\sigma = g + \sigma F$ random bit polynomial

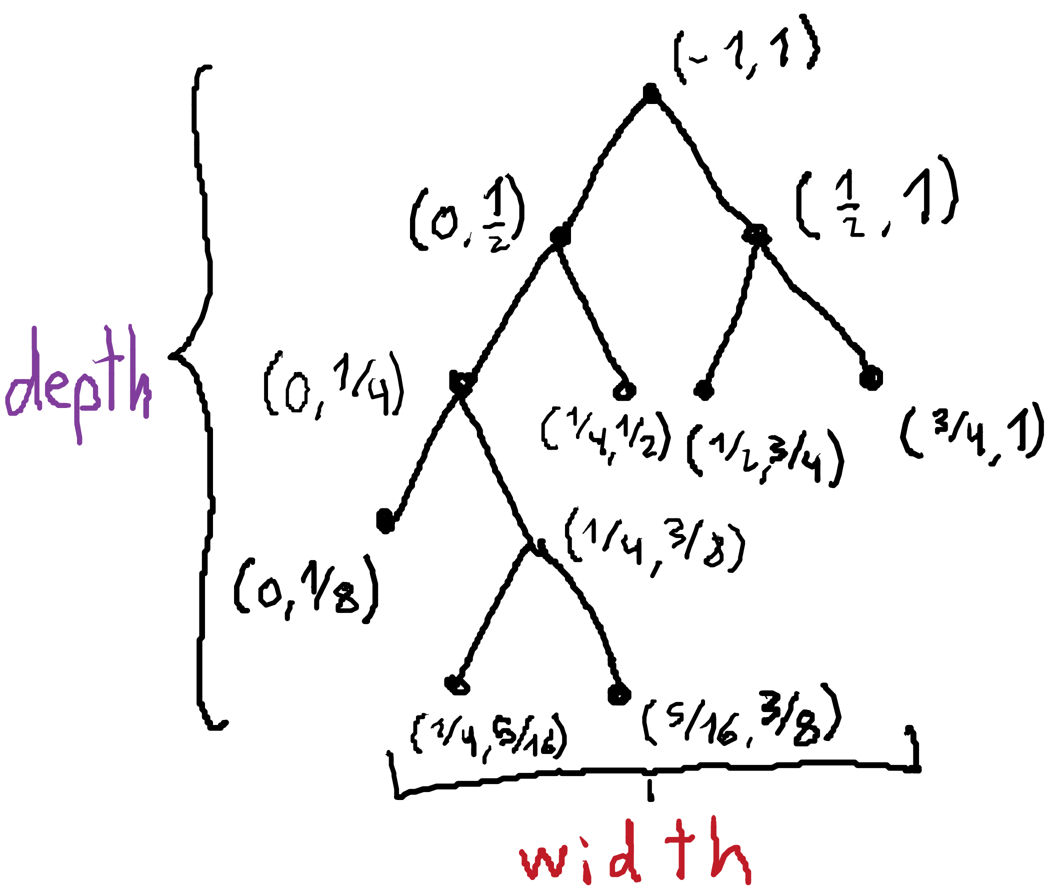
& $u(F_\sigma) \leq 1 + u(F) + \max\{\tau - \tau(F), \tau(\sigma)\}$

Lots of Details

will be Omitted

How to bound the run-time of DESCARTES?

$\mathcal{T}(g) \leftarrow$ DESCARTES' Computation Tree



PROP.

$$\mathrm{cost}(\text{DESCARTES}, g) \leq \mathcal{O}\big(d\tau\,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}\,\mathcal{T}(g) + d^2\,\mathrm{width}\,\mathcal{T}(g)\,\mathrm{depth}^2\,\mathcal{T}(g)\big)$$

I.e. size of $\mathcal{T}(g)$ bounds run-time of DESCARTES!

The Ingredients of the Analysis I: Condition Number

$$C(g) := \max_{x\in[-1,1]} \frac{\sum_{k=0}^{d} |g_k|}{\max\{|g(x)|, |g'(x)|/d\}}$$

$$C(g) = \infty \Leftrightarrow g \text{ has a singular root in } [-1,1]$$

$$\frac{1}{C(g)} \sim \begin{cases} \text{How much I have to perturb} \\ \quad \text{the coefficients of } g \text{ so that} \\ \quad\quad \tilde{g} \text{ has a singular root in } [-1,1] \end{cases}$$

Bounding depth $\mathcal{T}(g)$

Prop.

$$\text{depth } \mathcal{T}(g) \leq 5 + \log d + \log C(g)$$

$$\text{sep. between complex roots of } g \text{ nearby } [-1,1] \geq \frac{1}{12\, d\, C(g)}$$

Here no → $\not\wedge$

$$1/2^{\text{depth } \mathcal{T}(g)}$$

Here it is essencial for roots being near [-1,1]!

Bounding width $\tau(g)$

Prop

width $\tau(g)$

$\leq$ # complex roots of g nearby $[-1,1]$

Important for good bound of the RHS!

Secret tool to bound RHS:

Titchmarsh's thm.

The Ingredients of the Analysis II: Cont. Probabilistic Toolbox

We can handle

$C(f)$ (& # complex roots of f nearby $[-1,1]$)

for a wide class of random f

as long as continuous distribution

— using geometric functional analysis

... but we don't have a continuous dist.

The Ingredients of the Analysis III: Ball's Smoothing Trick

$x \in \mathbb{Z}^N$ discrete random variable

$y \in \mathbb{R}^N$ s.t. $y_i \sim \mathcal{U}((-\frac{1}{2}, \frac{1}{2}))$ i.i.d.

$\longrightarrow$ $x + y$ continuous random var.

We can use our old cont. toolbox!

SUMMING UP:

DESCARTES' SOLVER

IS ALMOST OPTIMAL ON AVERAGE!

... AND THAT'S WHY DESCARTES WORKS SO WELL

Muito Obrigado
pela Atenção!

¡Muchas Gracias
por su Atención!