AAA Online Seminar
TU Braunschweig
& U Osnabrück

New Condition-Based Bounds for the Number of Real Zeros

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Which questions are we interested in?

- Given a real polynomial system, how many real zeros does it have?

  what are the possibilities?

  what are the restrictions?

- Given a random real polynomial system, what are the statistics of its number of real zeros?

  how robust are our estimates?

  how do the estimates depend on the structures?

$\mathbb{R}$eal is the real complex, not $\mathbb{C}$omplex!

Let

$$f : \begin{cases} f_1 = \sum_{|\alpha| \leq d_1} f_{1,\alpha} X^\alpha \\ \vdots \\ f_n = \sum_{|\alpha| \leq d_n} f_{n,\alpha} X^\alpha \end{cases}$$

have generic complex / $\mathbb{R}$eal coefficients, then:

$$\# Z(f, \mathbb{R}^n) \leqslant \# Z(f, \mathbb{C}^n) = \mathcal{D} := \prod_{i=1}^{n} d_i$$

Bézout

$\mathbb{R}$eal is the real complex, not $\mathbb{C}$omplex II

Let $A_1, \ldots, A_n \subseteq \mathbb{N}^n$,

$$f : \begin{cases} f_1 = \sum_{\alpha \in A_1} f_{1,\alpha} X^\alpha \\ \vdots \\ f_n = \sum_{\alpha \in A_n} f_{n,\alpha} X^\alpha \end{cases}$$

have generic complex / $\mathbb{R}$eal coefficients,

then:

$$\# Z(f, (\mathbb{R}^*)^n) \leq \# Z(f, (\mathbb{C}^*)^n) = MV(\mathrm{conv}(A_1), \ldots, \mathrm{conv}(A_n))$$

Bernstein-Khovanskii-Kushnirenko

Real is the real complex, not Complex III

Kushnirenko Hypothesis III

THE HOLY GRAIL OF RAG

Let $A_1, \dots, A_n \subseteq \mathbb{N}^n$,

$$f: \left\{ f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha \right. \quad (i = 1, \dots, n)$$

Then:

$$\# Z_r(f, (\mathbb{R}^*)^n) \leq \mathrm{poly}\left(n, \sum_{i=1}^{n} \# A_i\right)^n$$

Widely open!

(n=1) $\# Z_r(f, \mathbb{R}^*) \leq 2 \# A_1$ (Descartes' rule of signs)

(n=2) ($A_2 = \{(\alpha_1,\alpha_2) \mid \alpha_1 + \alpha_2 \leq d_2\}$) $\# Z_r(f, (\mathbb{R}^*)^2) \leq O(\# A_1 d_2^3 + (\# A_1)^3 d_2^2)$
(Sevastyanov) (Koiran, Portier, Tavenas)

($A_2 = \{(0,0),(1,0),(0,1)\}$) $\# Z_r(f, (\mathbb{R}^*)^2) \leq 6t - 7$ (Bihan, El Hilany)

(A_2 arbitrary) Open!

Z_r: regular zeros

Real is the real complex, not Complex IV

Kushnirenko Hypothesis III

THE HOLY GRAIL OF $\mathbb{R}$AG

Let $A_1, \ldots, A_n \subseteq \mathbb{N}^n$,

$$f : \left\{ f_i = \sum_{\alpha \in A_i} f_{i,\alpha} X^\alpha \qquad (i = 1, \ldots, n) \right.$$

Then:

$$\#Z_r(f, (\mathbb{R}^*)^n) \le \mathrm{poly}\left(n, \sum_{i=1}^n \#A_i\right)^n$$

Widely open!

(General n) $\left(\#\left(\bigcup_{i=1}^n A_i\right) \le t\right)$ $\#Z_r(f, (\mathbb{R}^*)^n) \le O\left(2^{n+\binom{t-n-1}{2}} n^{t-n-1}\right)$

(Bihan, Sottile)

$(A_1 = \cdots = A_n, \#A_i \le t)$ $(f_{i,\alpha} \sim N(0, \sigma(\alpha))$ ind.$)$

$$\mathbb{E}_f \#Z_r(f, (\mathbb{R}^*)^n) \le 2\binom{t}{n} \le \frac{2t^n}{n!}$$

(Bürgisser, Ergür, TC)

Z_r: regular zeros

$\mathbb{R}$eal is the real complex, not $\mathbb{C}$omplex V

A Concrete Example

$$\mathcal{F}: \begin{cases} \alpha_1 + \beta_1 X + \gamma_1 Y + \delta_1 XYZ^d \\ \alpha_2 + \beta_2 X + \gamma_2 Y + \delta_2 XYZ^d \\ \alpha_3 + \beta_3 X + \gamma_3 Y + \delta_3 XYZ^d \end{cases}$$

For generic $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$,

$$\#Z(\mathcal{F}, \mathbb{C}^3) = d$$

LARGE!

For generic $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$,

$$\#Z(\mathcal{F}, \mathbb{R}^3) \leq 1$$

small!

A Random Path towards...

Kushnirenko Hypothesis III

Kushnirenko Random Hypothesis (Unmixed) Let $A \subseteq \mathbb{N}^n$ and a random real polynomial system

$$\mathfrak{f}: \left\{ \mathfrak{f}_i = \sum_{\alpha \in A} \mathfrak{f}_{i,\alpha} X^{\alpha} \quad (i = 1, \dots, n) \right.$$

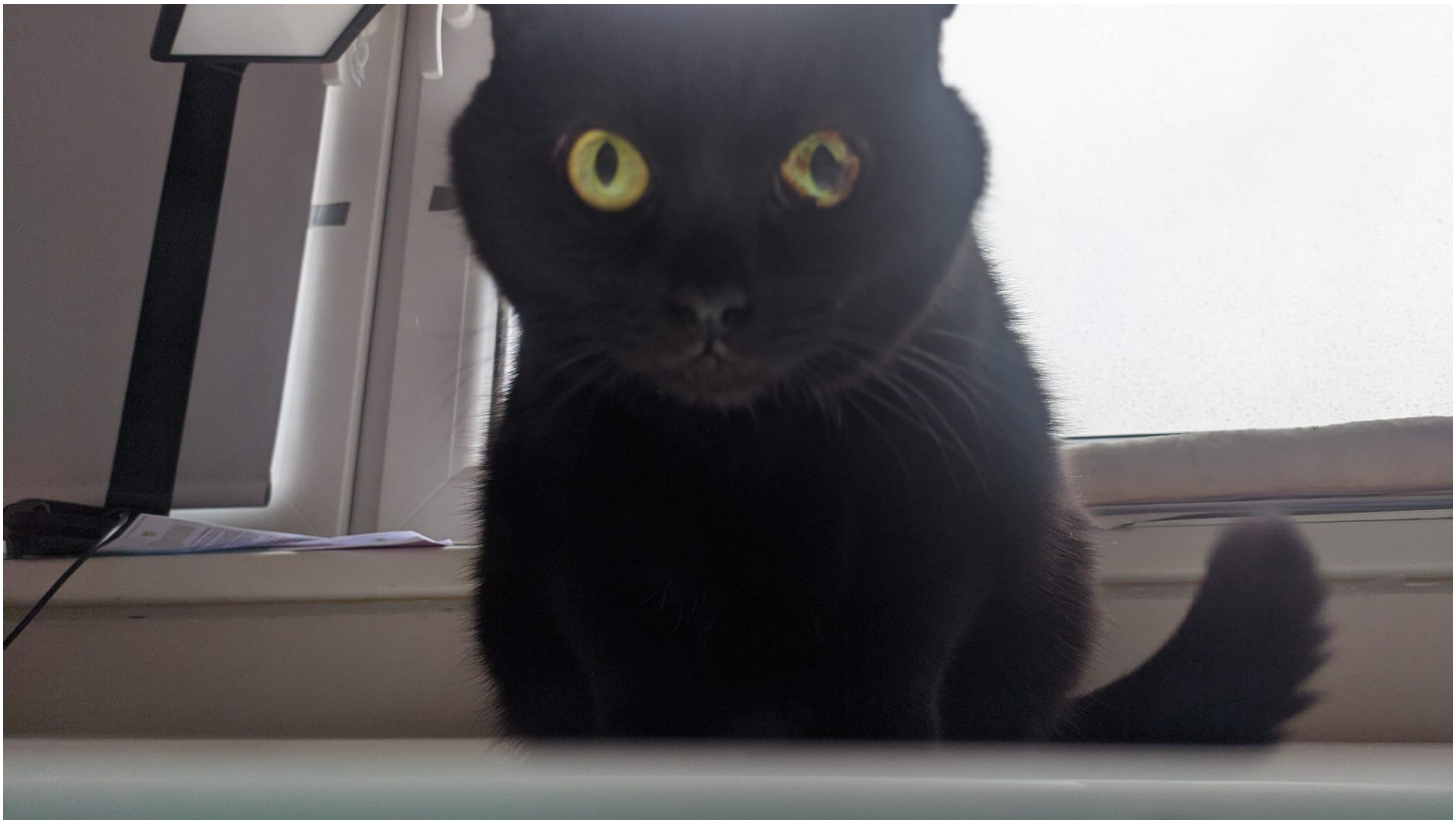
with the $\mathfrak{f}_{i,\alpha}$ random independent random variables, then for all $\ell \in \mathbb{N}$,

$$\mathbb{E}\, \#Z_r(\mathfrak{f}, \mathbb{R}^n_+)^{\ell} \leq \mathrm{poly}(n, \#A)^{n\ell}$$

- Equivalent to Kushnirenko Hypothesis III
- It creates a continuum between where we are and the Holy Grail of Real Algebraic Geometry

All this is very nice,

but do you have anything new to say?



Condition Numbers I

Def. Let $f \in \mathcal{P}_d[n] := \{ f \in \mathbb{R}[X_1, \dots, X_n]^n \mid \deg f_i \leq d_i \}$, and $x \in I^n := [-1,1]^n$, the local condition number of f at x is

$$C(f,x) := \frac{\|f\|_1}{\max\{\|f(x)\|_\infty, \sigma_n(\Delta^{-1} D_x f)\}}$$

where

$$\Delta := \operatorname{diag}(d_1, \dots, d_n)$$

$$\|f\|_1 := \max_i \sum_\alpha |f_{i,\alpha}|$$

$$\sigma_n(A) = \min_{v \neq 0} \frac{\|Av\|_\infty}{\|v\|_\infty} = \|A^{-1}\|_{\infty,\infty}^{-1}$$

Condition Numbers II

Def. Let $f \in \mathcal{P}_d[n] := \{ f \in \mathbb{R}[x_1, \dots, x_n]^n \mid \deg f_i \le d_i \}$, the condition number of f is

$$C(f) := \sup_{x \in I^n} C(f, x)$$

Condition Number Theorem. Let $f \in \mathcal{P}_d[n]$, then

$$\frac{\|f\|_1}{\mathrm{dist}_1(f, \Sigma)} \le C(f) \le n(D+1) \frac{\|f\|_1}{\mathrm{dist}_1(f, \Sigma)}$$

where $\Sigma := \{ g \in \mathcal{P}_d[n] \mid \text{every zero of } g \text{ in } I^n \text{ is regular} \}$

$C(f)$ measures how far is f from the discriminant.

The New Bound

Thm. Let $f \in \mathcal{P}_d[n]$. Then

$$\#Z(f, I^n) \leq \max\{\mathcal{O}(n\log(2D))^{2n}, \mathcal{O}(\log(2D)\log C(f))^n\}$$

Logarithmic!

Cor. Real polynomial systems with many zeros are ill-conditioned.

Proof Ideas I

1. Analytic approximation
+ Smale's α-theory

Real polynomial systems can be approximated locally by polynomial of degree

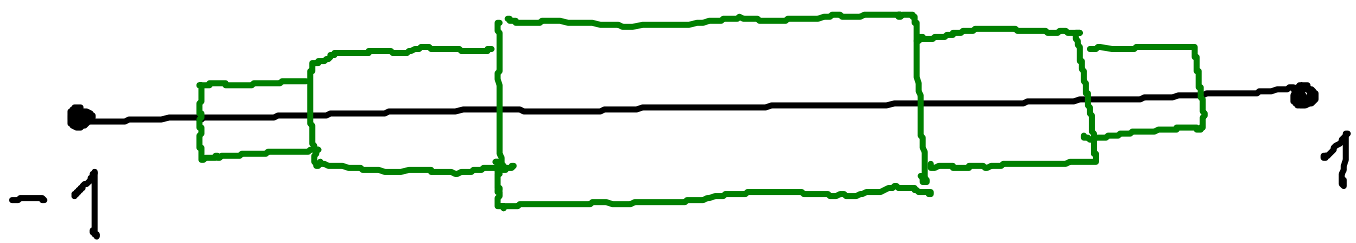
$$\max\{\mathcal{O}(\log D),\ \mathcal{O}(\log C(f))\}$$

with the same number of real zeros

Generalizes an idea by Moroz for the univariate case

Proof Ideas II

2. A cover where the local approximation happens can be easily constructed



This cover can be constructed a priori, there is computational content in the proof.

A similar bound
can be obtained
in the projective setting
for the orthogonally invariant
condition number of
Cucker, Krick,
Malajovich & Wschebor.

Random consequences I.

Def. A Kac random polynomial system with support A is a random system

$$\mathfrak{f} : \left\{ f_i = \sum_{\alpha \in A} \mathfrak{f}_{i,\alpha} X^\alpha \quad (i = 1, \dots, n) \right.$$

where the $\mathfrak{f}_{i,\alpha} \sim N(0,1)$ i.i.d.

⚠ Our probabilistic results hold also if $\mathfrak{f}_{i,\alpha} \sim \mathcal{U}([-1,1])$ or more general probability distributions.

Random consequences II

Thm. Let $f \in \mathcal{P}_d[n]$ be a random Kac system with support $A \ni 0$. Then

$$\mathbb{P}(\log C(f) \geq t) \leq \mathrm{poly}(n, D, \#A)^n e^{-ct}$$

for some universal constant $c > 0$.

⚠ Condition numbers are easier to control probabillistically

Cor. $\mathbb{E}\, C(f)^e \leq (\tilde{c}\, n^2 \log D \log \#A e)^e$

for some universal constant $\tilde{c} > 0$.

Random consequences III

Cor. $\mathbb{E}\#Z(F, I^n)^e \leq (\tilde{c}\, n^3 \log^2 D\ \#A\ e)^{en}$

for some universal constant $\tilde{c} > 0$.

Cor. Random real polynomials with many zeros are extremely rare.

Eskerrik Asko!

Galderak?
Oder Fragen?