Metric Restrictions on the Number of Real Zeros

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As some ideas are less technical in the setting of homogeneous polynomials, I will focus on that setting.

Symmetry makes life easier!

NOTATONS

- $X_0, X_1, \dots, X_n$ homogeneous variables
- $\mathcal{H}_d[n] := \prod_{i=1}^{n} \mathbb{R}[X_0, \dots, X_n]_{d_i}$
- For $f \in \mathcal{H}_d[n]$, and $x \in \mathbb{S}^n$
  - $\bar{D}^k_x f = \left(\frac{\partial^n f}{\partial X_{i_1} \cdots \partial X_{i_k}} \right)$
  - $D_x f = \bar{D}_x f \, (\mathbb{I} - x x^t) = \bar{D}_x f |_{T_x \mathbb{S}^n}$
  - Weyl norm $\|f\|_W := \sqrt{\sum_{i=1}^{n} \sum_{|\alpha| = d_i} \binom{d_i}{\alpha}^{-1} |f_{i,\alpha}|^2}$
  - $Z_{\mathbb{S}}(f) := \{ x \in \mathbb{S}^n \mid f(x) = 0 \}$

DISCRIMINANT CHAMBERS

$$\Sigma := \{ g \in \mathcal{H}_d[n] \mid \mathcal{Z}_S(g) \text{ singular} \}$$

↑ Here is where changes can occur!

Prop. $\mathcal{H}_d[n] \setminus \Sigma \ni g \mapsto \# \mathcal{Z}_S(g)$ locally constant.

Def. A <u>discriminant chamber</u> $\mathcal{A}$ is a connected component of $\mathcal{H}_d[n] \setminus \Sigma$

Question. Given $f \in \mathcal{H}_d[n]$ KSS* random polynomial system, what is $\mathbb{P}(g \in \mathcal{A})$?

*Also for dobro

Interlude: Random Polynomial System

Let $f \in \mathcal{H}_d[n]$ with

$$f_i = \sum_\alpha \sqrt{\binom{d_i}{\alpha}}\, c_{i,\alpha}\, X^\alpha$$

be random.

- **KSS:** $c_{i,\alpha}$ i.i.d. normal
- **Dobro:** $c_{i,\alpha}$ independent, anti-conc + subgausian
- **EPR:** f_i independent, $f_i(x)$ anti-conc. + subgaus.

Ergür, Paouris
Rojas

CONDITION NUMBER

Def. Given $f \in \mathcal{H}_d[n]$, the <u>condition number</u> of f is

$$\kappa(f) := \max_{x \in S^n} \frac{\|f\|_W}{\sqrt{\|f(x)\|^2 + \|D_x f^{-1} \Delta\|^{-1}}} \in [1, \infty]$$

where $\Delta := \mathrm{diag}(d_1, \ldots, d_n)$

THM (Condition Number Theorem) [Cucker, Krick, Malajovich, Wschebor]

Let $f \in \mathcal{H}_d[n]$, then

$$\kappa(f) = \frac{\|f\|_W}{\mathrm{dist}_W(f, \Sigma)}$$

κ is a metric discriminant!

Inradius of a Disc. Chamber

Def. Let $\mathcal{A} \subseteq \mathcal{H}_d[n] \setminus \Sigma$ be a disc. chamber, and consider

$$\kappa(\mathcal{A}) := \min\{\kappa(g) \mid g \in \mathcal{A}\}$$

Obs. Given $g \in \mathcal{A}$,

$$g + \|g\|_W r B_W \subseteq \mathcal{A} \iff \operatorname{dist}(g, \Sigma) \geq \|g\|_W r$$

Prop. $1/\kappa(\mathcal{A})$ is the inradius of $\mathcal{A}$,

i.e., $$1/\kappa(\mathcal{A}) = \max\{r \mid \exists g \in \mathcal{A} : B_W(g, r\|g\|_W) \subseteq \mathcal{A}\}$$

↑ $\mathcal{A}$ is conic!

$B_W := \{g \mid \|g\|_W < 1\}$, $B_W(g,s) := \{h \mid \|h-g\|_W < s\}$

BOUNDING PROBABILITIES I

Let $f \in \mathcal{H}_d[n]$ be a random KSS pol. system, then:

$$\mathbb{P}(f \in \mathcal{A}) \leq \mathbb{P}(\mathcal{K}(f) \geq \mathcal{K}(\mathcal{A}))$$

$$\leq 32 D^2 \mathcal{D}^{1/2} N^{1/2} n^3 \frac{\ln^{1/2} \mathcal{K}(\mathcal{A})}{\mathcal{K}(\mathcal{A})}$$

Cucker, Krick, Malajovich, Wschebor

Can we lower bound $\mathcal{K}(\mathcal{A})$? Yes!

The above works for very general random assumptions...

AN UNEXPECTED INEQUALITY

THM. Let $f \in \mathcal{H}_d[n]$. Then

$$\#\mathcal{Z}_S(f) \leq c^n D^{n/2} \log^n \kappa(f)$$

where $c > 1$ is universal.

COR. Let $f \in \mathcal{H}_d[n]$. Then

$$\kappa(f) \geq 2^{\frac{\#\mathcal{Z}_S(f)^{1/n}}{c\, D^{1/2}}}$$

COR. If $f \in \mathcal{H}_d[n]$ has many real zeros, then f is ill-conditioned!

Bounding Probabilities II

Thm. Let $\mathcal{A} \subseteq \mathcal{H}_d[n]/\Sigma$ be a discriminant chamber and $N(\mathcal{A})$ the number of real zeros of any system in $\mathcal{A}$, then:

$$\mathbb{P}(f \in \mathcal{A}) \leq 2^{a n \log D - b \frac{N(\mathcal{A})^{1/n}}{D^{1/2}}}$$

where $a, b > 0$ are universal.

Cor. Disc. chambers with systems with many real zeros are always small.

Bounding Probabilities III

THM. Let F be a KSS random polynomial system, then

$$\# \mathcal{Z}_S(F)^{1/n}$$

is subexponential with constant

$$a\, n\, D^{1/2} \log D$$

where $a > 0$ is universal. I.e. for $e \geq 1$,

$$\left(\mathbb{E}\, \mathcal{Z}_S(F)^e \right)^{1/e} \leq a^n\, n^{2n}\, D^{n/2} \log^n D\; e^n$$

WHAT'S BEHIND THE UNEXPECTED INEQUALITY?

(MOROZ 2021)
To solve a univariate polynomial uses many extremely low degree approximations based on Taylor expansions.

We generalize this to higher dimensions

THM. Let $\mathfrak{f} \in \mathcal{H}_d[n]$, and $r < 1/aD^{1/2}$. Then for all $x \in \mathbb{S}^n$, $\mathfrak{f}|B_{\mathbb{S}}(x,r)$ can be approximated by a $\mathcal{O}(\log \mathcal{K}(\mathfrak{f}))$-degree pol. system with zeros that approximate à la Smale all those of $\mathfrak{f}$ in $B_{\mathbb{S}}(x,r)$

OBS. This many extremely low-degree approx. scheme differs from the one low-degree of Diatta & Lerario.

OTHER CASES...

- Kac random polynomial systems
- Underdetermined polynomial systems (only volume for now)
- Sparse Kac random polynomial systems (we have to see how strong can our results be)

Future

Can we have algorithms working in time that it's bounded by

$$\log^n K(g)$$

and not $K(g)^n$?

This should give very fast algorithms in N~~R~~AG

Main obstacle: Avoid computing $K(g)$ directly...

¡Muchas gracias por su atención!