Effective Algebraic Geometry Seminar

Summer Semester 2018. Thursdays, 16:15-17:45; MA316.

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1. Description of the seminar

Modern algebraic geometry has developed a deep and far reaching theory to understand the zero sets of polynomial equations. Today's immense computational power brought the challenge of transfering the wisdom of algebraic geometry into working algorithms. In this seminar, we strive to prove theorems that can be communicated to computers. In other words, we develop effective methods in algebraic geometry.

In order to solve computational problems in algebraic geometry, there are usually two kinds of methods: symbolic and numerical. On the one hand, symbolic algorithms are always guaranteed to work, can be proven to halt in finitely many steps, but usually suffer from high computational complexity; on the other hand, numerical methods don't always work, are not guaranteed to halt in finitely many steps, but are provably fast for generic inputs. It is said that hybrid numeric-symbolic approaches are marriage made in heaven. However, it still remains a challenge today to combine symbolic and numerical approaches in computational algebraic geometry.

In this seminar, we will learn about both the symbolic and numerical methods, with particular emphasis on Gröbner basis, resultants, and Kaltofen's algorithm on the symbolic side; Newton's method, Smale's alpha theory, and homotopy continuation methods on the numerical side.

2. Plan for the seminar

2.0. Introduction (18/04) Topics of the seminar will be introduced and they will be distributed among attending students. The scheduled time of the seminar will be discussed.

2.1. Gröbner basis I: Concept (03/05) [Kathlén Kohn] Euclidean division. Monomial ordering. Examples of monomial orderings. Euclidean division by several polynomials. Definition of Gröbner basis. Characterization of Gröbner basis in terms of monomial ideals. [CLO07, 2.1-5]

2.2. Gröbner basis II: Computation and application (17/05) [Florian Blatt] Buchberger's criterion and algorithm. Applications. Elimination theory. [CLO07, 2.6-8,3.1-2]

2.3. Resultants: Concept and computation (24/05) [Mario Kummer] Resultant of univariate polynomial. Resultant of multivariate polynomials. Elimination theory. [CLO07, 3.5-6]

2.4. Algorithms in Algebra and PIT I: Framework (31/05) [Josue Tonelli-Cueto] Computational presentation of polynomials. Randomization. Schwartz-Zippel Lemma [AB09, Lemma A.36]. [Sab05] [AB09, Chapters 7 and 16]

2.5. Algorithms in Algebra and PIT II: Factorization of polynomials I (14/06) [Alperen Ergür] Kaltofen's algorithm through PIT. Reduction to bivariate and univariate case. [KSS15, Tim15]

2.6. Algorithms in Algebra and PIT III: Factorization of polynomials II (28/06) [Sascha Timme] Kaltofen's algorithm through PIT: Solution of univariate case. Hensel's lifting. [KSS15, Tim15]

2.7. Newton's method: Smale's alpha theory (05/07) [Oğuzhan Yürük] Newton's method. Smale's α, β and γ parameters. Statement of Smale's γ-theorem. Statement and proof of Smale's α-theorem. [Mal11, Chapter 7]

2.8. Homotopy Continuation I: Introduction (05/07) [Henning Seidler] Introduce the basic strategy of homotopy continuation. Show several examples of types of homotopy continuations without any analysis. [BC13, 15.1] [SVW05]

2.9. Geometric Framework for Condition (12/07) [Levent Doğan] Definition of condition number. State Proposition 14.1 without proof. Section 14.1.1 in detail, emphasizing intuition. Section 14.1.2 and 14.3 in detail. Section 14.3.1 showing how the theory is applied. [BC13, Chapter 14]

2.10. Homotopy Continuation II: Dense framework and Condition Number Theorem (Cancelled) Section 16.1, 16.2 and 16.4 are a must. Describe Projevtive Newton's Method, section 16.6. [BC13, Chapter 16]

2.11. Homotopy Continuation III: Adaptative Linear Homotopy (19/07) [Peter Bürgisser] Describe the adaptative linear homotopy. Deatailed proof of 17.3. Description of Algorithm 17.4. [BC13, Chapter 17]

2.12. Homotopy Continuation IV: Beltrán-Pardo Randomization (Cancelled) Section 17.6. [BC13, Chapter 17]

3. Enrollment procedure and evaluation for BMS students

For taking part in this seminar, please write an email to all of us

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with the subject line "EAG Seminar". Please indicate in your email whether if you are willing to receive a grade from this seminar (in case of a BMS student), and if you have a preferred topic to present. We can not promise to match everyone's best topic, but we will try to accommodate preferences as much as possible.

We encourage every participating student to write lecture notes for their presentation. We hope these notes will accumulate into a coherent whole.

For BMS students receiving a grade from this seminar, it is compulsary to write the lecture notes. Grade assesment for BMS students will be based on a comprehensive evaluation of these notes together with the oral presentation.

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Bibliography

[AB09] Sanjeev Arora and Boaz Barak. *Computational complexity. A modern approach*. Cambridge University Press, Cambridge, 2009.

[BC13] Peter Bürgisser and Felipe Cucker. *Condition: The geometry of numerical algorithms*, volume 349 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2013.

[CLO07] David Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007.

[Ded06] Jean-Pierre Dedieu. *Points fixes, zéros et la méthode de Newton*, volume 54 of *Mathématiques & Applications (Berlin)*. Springer, Berlin, 2006.

[KSS15] Swastik Kopparty, Shubhangi Saraf, and Amir Shpilka. Equivalence of polynomial identity testing and polynomial factorization. *Comput. Complexity*, 24(2):295–331, 2015. Available at `https://link.springer.com/article/10.1007/s00037-015-0102-y`.

[Mal11] Gregorio Malajovich. *Nonlinear equations*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2011. 28° Colóquio Brasileiro de Matemática. [28th Brazilian Mathematics Colloquium] Available at `http://www.labma.ufrj.br/~gregorio/nonlinear/nonlinear.pdf`.

[Sab05] Juan Sabia. Algorithms and their complexities. In *Solving polynomial equations*, volume 14 of *Algorithms Comput. Math.*, pages 241–268. Springer, Berlin, 2005.

[SVW05] Andrew J. Sommese, Jan Verschelde, and Charles W. Wampler. Introduction to numerical algebraic geometry. In *Solving polynomial equations*, volume 14 of *Algorithms Comput. Math.*, pages 301–335. Springer, Berlin, 2005.

[Tim15] Sascha Timme. *Black box factorization of multivariate polynomials*. Bachelor Thesis at Technische Universität Berlin. 2015. Available at `http://www3.math.tu-berlin.de/algebra/teaching/theses/2015-bsc-sascha-timme.pdf`.