# Plantinga-Vegter algorithm takes average polynomial time

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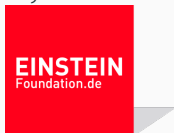Slides at https://tonellicueto.xyz/pdf/ISSAC19_slides.pdf

This presentation is about the accepted paper

*Plantinga-Vegter algorithm takes average polynomial time*

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# Plantinga-Vegter algorithm

## Setting

What do we have?

- An implicit curve $C$ inside $[-a, a]^2$
  given by a $C^1$ function $f : [-a, a]^2 \to \mathbb{R}$
- Interval approximations $\Box f$ of $f$ and $\Box \partial f$ of $\partial f$

What do we want?

- Piecewise-linear approximation $L$ of $C$ in $[-a, a]^2$ such that
  $([-a, a]^2, C)$ and $([-a, a]^2, L)$ are isotopic

Any assumptions?

- $C$ smooth
- $C$ Intersects the boundary of $[-a, a]^2$ transversely

**Algorithm:** PV Algorithm for curves (Plantinga, Vegter; 2004)

**Input:** $a \in (0, \infty)$ and $f : \mathbb{R}^2 \to \mathbb{R}$
with interval approximations $\square[f]$ and $\langle \square[\partial f], \square[\partial f] \rangle$

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SUBDIVISION:
Starting with the trivial subdivision $\mathcal{S} := \{[-a, a]^n\}$, repeatedly
 subdivide each $J \in \mathcal{S}$ into 4 squares until for all $J \in \mathcal{S}$,

$$0 \notin \square f(J) \text{ or } 0 \notin \langle \square \partial f(J), \square \partial f(J) \rangle$$
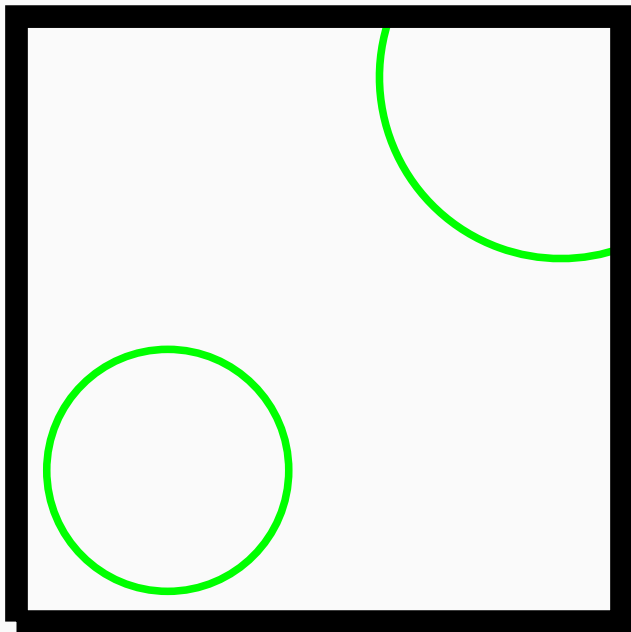
CONSTRUCTION:
Construct piecewise-linear curve $L$
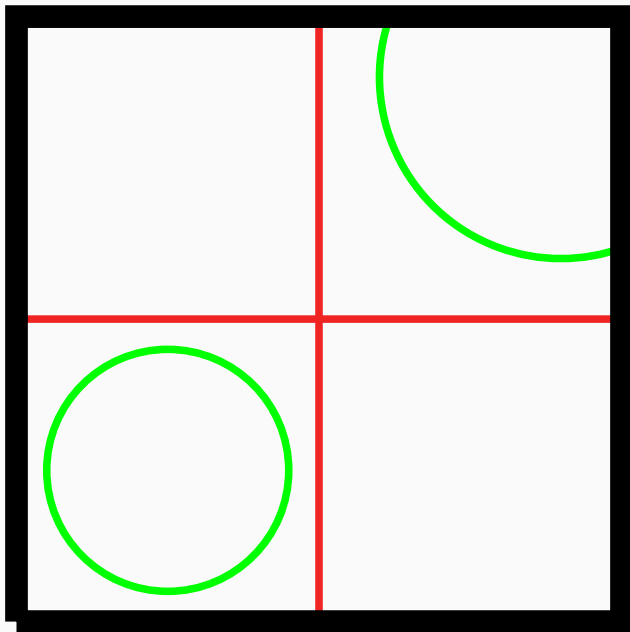joining the midpoints of "small" edges of each $J \in \mathcal{S}$ with oposite
 $f$-signs at their vertices

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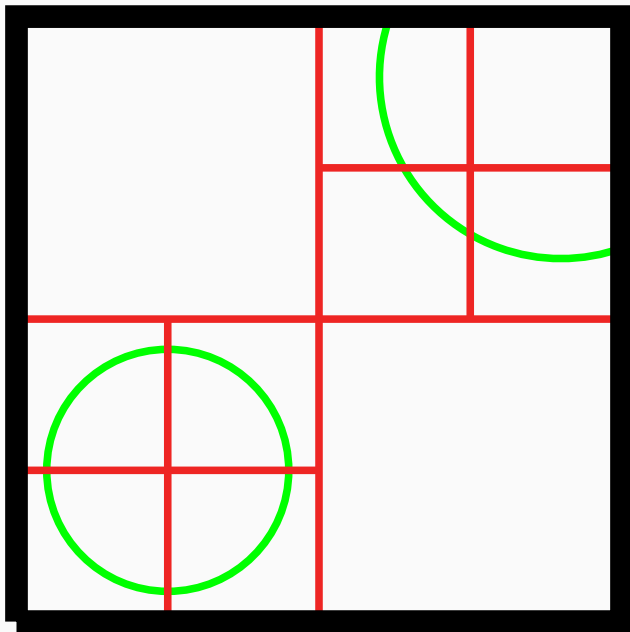**Output:** Piecewise-linear approximation $L$ of $C = f^{-1}(0) \cap [-a, a]^2$
        isotopic to it
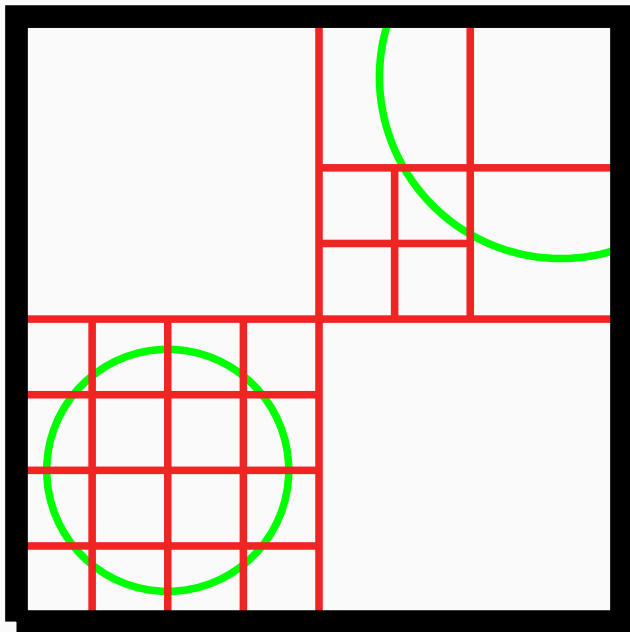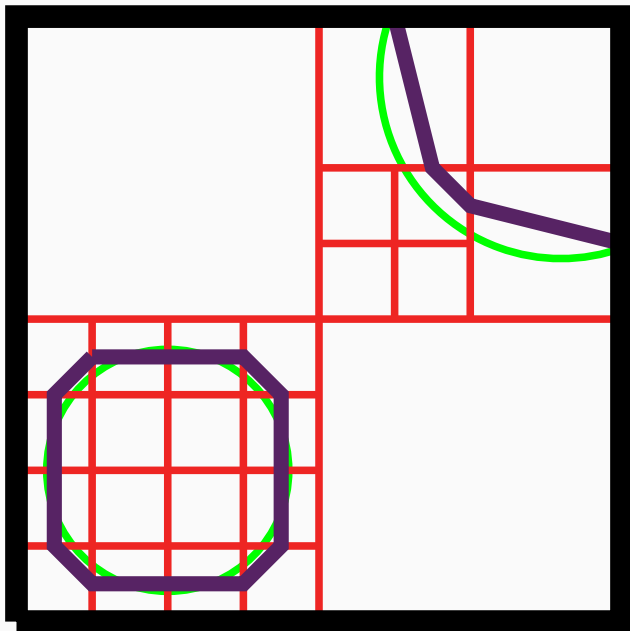
1. Plantinga-Vegter algorithm can be generalized to produce isotopic approximations of surfaces (Plantinga, Vegter; 2004)
   This is really why is called Plantinga-Vegter!
   Very efficient in practice

2. The subdivision method can be generalized to higher dimensions (Burr, Gao, Tsigaridas; ISSAC2017)
   but no construction of the piecewise-linear approximation...

We will focus on the later, since...

complexity of the algorithm is mainly that of the subdivision part

We will mainly count the number of subdivisions, because...

$$\text{cost(subdivision algorithm)} \sim \#(\text{subdivisions}) \cdot \text{cost(evaluations)}$$

**Algorithm:** Subdivision of PV Algorithm (Burr, Gao, Tsigaridas; ISSAC2017)

**Input:** $a \in (0, \infty), f : \mathbb{R}^n \to \mathbb{R}$
with interval approximations $\square[hf]$ and $\square[h'\partial f]$
for some functions $h, h' : \mathbb{R}^n \to (0, \infty)$

Starting with the trivial subdivision $\mathcal{S} := \{[-a, a]^n\}$, repeatedly
subdivide each $J \in \mathcal{S}$ into $2^n$ cubes until the condition

$$C_f(J) : 0 \notin \square[hf](J) \text{ or } 0 \notin \langle \square[h'\partial f], \square[h'\partial f] \rangle$$

holds for all $J \in \mathcal{S}$

**Output:** Subdivision $\mathcal{S} \subseteq \mathcal{I}_n$ of $[-a, a]^n$
such that for all $J \in \mathcal{S}$, $C_f(J)$ is true

$h, h'$ depend on the setting and the interval arithmetic one uses

Can we understand the complexity of the subdivision of the PV algorithm for polynomials $f \in \mathcal{P}_{n,d}$ in terms of the number of variables $n$ and the degree $d$?

Local size bounds and
continuous amortization
(Previous state-of-the-art)

**Idea**: Give a bound to the size of the smallest box containing a point not satisfying the condition.

**Definition (Burr, Gao, Tsigaridas; ISSAC2017)**

A *local size bound* for $f$ is a function $b_f : \mathbb{R}^n \to [0, \infty)$ such that for all $x \in \mathbb{R}^n$,

$$b_f(x) \leq \inf\{\operatorname{vol}(J) \mid x \in J \in \mathcal{I}_n \text{ and } C_f(J) \text{ false}\},$$

where $C_f(J) : 0 \notin \square[hf](J)$ or $0 \notin \langle \square[h'\partial f], \square[h'\partial f] \rangle$

## 1st bound in terms of the local size bound

**Idea**: Boxes cover the cube. The volume of the boxes should add to that of the cube. Volume of the boxes is at least number of boxes times volume of smallest box.

### Proposition (Burr, Gao, Tsigaridas; ISSAC2017)

*The number of n-cubes of the final subdivision of the subdivision of the PV algoirhtm on input $(f, a)$, regardless of how the subdivision step is done, is at most*

$$(2a)^n / \inf\{b_f(x) \mid x \in [-a, a]^n\} \qquad \square$$

*where $b_f$ is a local size bound for f.*

(Burr, Gao, Tsigaridas; ISSAC2017) construct a local size bound for an integer polynomial $f \in \mathcal{P}_{n,d}$ and obtain the bound

$$2^{\mathcal{O}(nd^{n+1}(n\tau + nd \log(nd) + 9n + d) \log a)}$$

where $\tau$ is the bit-size of the coefficients.

**Refinement of the idea**: Not all boxes have the same size. "$\frac{\mathrm{d}x}{b_f(x)}$ is, up to constant, the infinitesimal number of boxes needed to cover $x$." Formalized by (Burr; 2016) using the technique known as *continuous amortization* introduced in (Burr, Krahmer, Yap; 2009)

**Theorem (Burr, Gao, Tsigaridas; ISSAC2017)**
*The number of n-cubes of the final subdivision of the PV algorithm on input $(f, a)$ is at most*

$$\max\left\{1, \int_{[-a,a]^n} \frac{2^n}{b_f(x)} \,\mathrm{d}x\right\}$$

*where $b_f$ is a local size bound for f. Moreover, the bound is finite if and only if the algorithm terminates.* □

Techniques of (Burr, Gao, Tsigaridas; ISSAC2017) cannot exploit it!

## Can we exploit it?

(Burr, Gao, Tsigaridas; ISSAC2017) said...

> *Even though our bounds are optimal, in practice, these are quite pessimistic [...]*

and

> *Since the complexity of the algorithm can be exponential in the inputs [size], the integral must be described in terms of additional geometric and intrinsic parameters.*

What can be a solution to these issues?

### Condition numbers*

*as described in the book *Condition* (Bürgisser, Cucker; 2013)

# Condition-based complexity

## Definition (Cucker)

Given an homogeneous polynomial $F \in \mathcal{H}_{n,d}$, the *local condition number* of $F$ at $y \in \mathbb{S}^n$ is

$$\kappa(F, y) := \frac{\|F\|}{\sqrt{F(y)^2 + \|\partial_y F_{|\mathrm{T}_y \mathbb{S}^n}\|^2 / d}}$$

where $\|F\|$ is the Weyl norm of $F$.
Given $f \in \mathcal{P}_{n,d}$, the *local affine condition number* of $f$ at $x \in \mathbb{R}^n$ is

$$\kappa_{\mathrm{aff}}(f, x) := \kappa(f^{\mathrm{h}}, \phi(x))$$

where $f^{\mathrm{h}}$ is the homogeneization of $f$ and $\phi : x \mapsto \dfrac{1}{\sqrt{1 + \|x\|^2}} \begin{pmatrix} 1 \\ x \end{pmatrix}$.

Unfortunately the theory is developed for the homogenous setting. Too many translations!

# Geometry of the local condition number

## Observation

$\kappa_{\text{aff}}(f, x) = \infty$ iff $V_{\mathbb{R}}(f)$ has a singularity at $x$

Ergo $\kappa_{\text{aff}}(f, x)$ says how near $V_{\mathbb{R}}(f)$ is of having a singularity at $x$. Can we be more concrete? YES!

## Theorem (Condition Number Theorem)

*Let $x \in \mathbb{R}^n$ and*

$$\Sigma_x := \{g \in \mathcal{P}_{n,d} \mid g(x) = \partial g(x) = 0\},$$

*i.e., $\Sigma_x$ is the set of hypersurfaces having $x$ as a singular point. Then for every $f \in \mathcal{P}_{n,d}$,*

$$\frac{\|f\|}{\kappa_{\text{aff}}(f, x)} = \text{dist}(f, \Sigma_x)$$

*where $\| \cdot \|$ is the Weyl norm of $\mathcal{P}_{n,d}$ and the distance is the induced by the Weyl norm of $\mathcal{P}_{n,d}$.* $\square$

The wanted "additional geometric and intrinsic parameter"

**Theorem (Cucker, Ergür, T.-C.; ISSAC2019)**

*Let $f \in \mathcal{P}_{n,d}$. Then*

$$x \mapsto 1 / \left( 2^{5/2} dn\kappa_{\mathrm{aff}}(f,x) \right)^n$$

*is a local size bound for f with the interval approximation of (Cucker, Ergür, T.-C.; ISSAC2019), and*

$$x \mapsto 1 / \left( 2^{3n} d^2 \kappa_{\mathrm{aff}}(f,x) \right)^n$$

*with the interval approximation of Remark 2.2. of (Burr, Tsigaridas, Yap; ISSAC2017)* □

**Theorem (Cucker, Ergür, T.-C.; ISSAC2019)**

*The number of n-cubes in the final subdivision of the subdision of the* PV *algorithm on input* $(f, a)$ *is at most*

$$d^n \max\{1, a^n\} 2^{n \log n + 9n/2} \, \mathbb{E}_{x \in [-a,a]^n} \left( \kappa_{\mathrm{aff}}(f, x)^n \right)$$

*if the interval approximation is that of (Cucker, Ergür, T.-C.; ISSAC2019), and at most*

$$d^{2n} \max\{1, a^n\} 2^{3n^2 + 2n} \, \mathbb{E}_{x \in [-a,a]^n} \left( \kappa_{\mathrm{aff}}(f, x)^n \right)$$

*if the interval approximation is that of (Burr, Tsigaridas, Yap; ISSAC2017).* □

Observe that the complexity depends on $\mathbb{E}_{x \in [-a,a]^n} \left( \kappa_{\mathrm{aff}}(f, x)^n \right)$ which varies with $f$. This is why the name condition-based complexity

# Probabilistic complexity

**Worst-case complexity analysis**:

*What is the worst running time?*

**Average complexity analysis**:

*What is the expectation of the running time on a random input?*

**Smoothed complexity analysis**: (Spielman, Teng; 2002)

*What is the worst running time after perturbing the input with a random perturbation (with weight $\sigma$)?*

Smoothed lies between worst-case and average complexity

- $\sigma \to 0$: We recover worst-case complexity
- $\sigma \to \infty$: We recover average analysis

### Definition (Cucker, Ergür, T.C.; ISSAC2019)

A *dobro random polynomial* $f \in \mathcal{H}_{n,d}$ with parameters $K$ and $\rho$ is a polynomial

$$f := \sum_{|\alpha|=d} \binom{d}{\alpha}^{1/2} c_\alpha X^\alpha$$

such that the $c_\alpha$ are independent random variables such that

P1 $\mathbb{E}c_\alpha = 0$ (centered),

P2 $(\mathbb{E}|c_\alpha|^p)^{\frac{1}{p}} \leq K\sqrt{p}$ for $p \geq 1$ (subgaussian with $\Psi_2$-norm $\leq K$), and

P3 $\max_{u \in \mathbb{R}}\{\mathbb{P}(|c_\alpha - u| \leq \varepsilon)\} \leq \rho\varepsilon$ (anti-concentration with constant $\rho$).

A *dobro random polynomial* $f \in \mathcal{P}_{n,d}$ is a polynomial $f$ such that its homogenization $f^h$ is so.

Examples of dobro random polynomials:

N *KSS random polynomial*:                    (KSS=Kostlan-Smale-Shub)
- $c_\alpha$ is Gaussian with unit variance
- $K\rho = 1/\sqrt{2\pi}$

U *Weyl random polynomial*:
- $c_\alpha$ is uniform distribution in $[-1, 1]$
- $K\rho \leq 1$

E A *p-random polynomial*:
- $c_\alpha$ has density function $\delta_p e^{-|t|^p}$ where $\delta_p$ being the appropriate constant and $p \geq 2$

Note: Dobro (добро) is a Russian adjective derived from добрый (dóbryj) which means kind, good, genial, gentle, soft, etc.

**Theorem (Cucker, Ergür, T.C.; ISSAC2019)**

*Let $f \in \mathcal{P}_{n,d}$ be a dobro random polynomial with parameters $K$ and $\rho$. The expected number of n-cubes in the final subdivision of the* PV *algorithm on input $(f, a)$ is at most*

$$d^{\frac{n^2+3n}{2}} \max\{1, a^n\} 2^{\frac{n^2+16n \log(n)}{2}} (c_1 c_2 K \rho)^{n+1}$$

*if the interval approximations is as in (Cucker, Ergür, T.C.; ISSAC2019) and*

$$d^{\frac{n^2+5n}{2}} \max\{1, a^n\} 2^{\frac{7n^2+9n \log(n)}{2}} (c_1 c_2 K \rho)^{n+1}$$

*if the interval approximation is as in (Burr, Gao, Tsigaridas; ISSAC2017). $c_1$ and $c_2$ universal constants.* □

With an improvement of our condition-based techniques we can eliminate the $n^2$ of the exponent!

**Theorem (Cucker, Ergür, T.C.; ISSAC2019)**
*Let $f \in \mathcal{P}_{n,d}$, $\sigma > 0$, and $g \in \mathcal{P}_{n,d}$ a dobro random polynomial with parameters $K$ and $\rho$. Then the expected number of n-cubes of the final subdivision of the PV algorithm for input $(q_\sigma, a)$ where $q_\sigma = f + \sigma\|f\|g$ is at most*

$$d^{\frac{n^2+3n}{2}} \max\{1, a^n\} 2^{\frac{n^2+16n\log(n)}{2}} (c_1 c_2 K\rho)^{n+1} \left(1 + \frac{1}{\sigma}\right)^{n+1}$$

*if the interval approximations (Cucker, Ergür, T.C.; ISSAC2019) and*

$$d^{\frac{n^2+5n}{2}} \max\{1, a^n\} 2^{\frac{7n^2+9n\log(n)}{2}} (c_1 c_2 K\rho)^{n+1} \left(1 + \frac{1}{\sigma}\right)^{n+1}$$

*if the interval approximation is as in (Burr, Gao, Tsigaridas; ISSAC2017). $c_1$ and $c_2$ are universal constants.* □

With an improvement of our condition-based techniques we can eliminate the $n^2$ of the exponent!

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$$\mathcal{O}\left(d^5\right) \text{ and } \mathcal{O}\left(d^6\right)$$

with the interval arithmetic of, respectively, (Cucker, Ergür, T.C.; ISSAC2019) and (Burr, Gao, Tsigaridas; ISSAC2017)

$$\mathcal{O}\left(d^{5/2}\log^{5/2} d\right) \text{ and } \mathcal{O}\left(d^3\log^3 d\right)$$

with the interval arithmetic of, respectively, (Cucker, Ergür, T.C.; ISSAC2019) and (Burr, Gao, Tsigaridas; ISSAC2017) With the new techniques! Compare to bit-complexity

$$\tilde{\mathcal{O}}(d^5\tau + d^6)$$

of the deterministic algorithm of (Kobel, Sagraloff; 2015) and (Diatta, Diatta, Rouillier, Roy, Sagraloff; 2018)

*Some comments*:

- Difference should be careful, complexity measured in different ways! But still, why so efficient?!
- PV algorithm does not work with singular curves, although there is work in this direction (Burr, Choi, Galehouse, Yap; 2012)
- Can we develop an hybrid approach working on all inputs with the same worst case, but faster on average?

Bere arretagatik eskerrik asko!
**谢谢大家**


Galderak?
**以下是提问时间**

# Idea for the condition-based bound

### Proposition

*Let $f \in \mathcal{P}_{n,d}$ and $x \in \mathbb{R}^n$. Then either*

$$|\widehat{f}(x)| > \frac{1}{2\sqrt{2d}\,\kappa_{\text{aff}}(f,x)} \quad \text{or} \quad \left\|\widehat{\partial f}(x)\right\| > \frac{1}{2\sqrt{2d}\,\kappa_{\text{aff}}(f,x)}.$$

Above, $\widehat{f}$ is given by

$$\widehat{f} : x \mapsto f(x)/\left(\|f\|(1+\|x\|^2)^{(d-1)/2}\right)$$

and $\widehat{\partial f}$ by

$$\widehat{\partial f} : x \mapsto \partial f(x)/\left(d\|f\|(1+\|x\|^2)^{d/2-1}\right)$$

being both functions $(1+\sqrt{d})$-Lipschitz.

# Ideas for the probabilistic bound

1. By Fubini-Tonelli, to bound

$$\mathbb{E}_f \mathbb{E}_{x \in [-a,a]^n} \kappa_{\text{aff}}(f,x)^n,$$

   it is enough to get tail bounds of $\kappa_{\text{aff}}(f,x)$ with respect to $f$ independently of $x$

2. Since $\Sigma_x$ is a linear subspace, we can write

$$\kappa_{\text{aff}}(f,x) = \frac{\|f\|}{\|P_x f\|}$$

   where $P_x : \mathcal{P}_{n,d} \to \Sigma_x^{\perp}$ is an orthogonal projection with respect the Weyl inner product.

3. Probabilistic steps for average analysis:
   3.1 $\kappa_{\text{aff}}(f,x)$ large implies either $\|f\|$ large or $\|P_x f\|$ small
   3.2 "$\|f\|$ large" controlled by subgaussian property
   3.3 "$\|P_x f\|$ small" controlled by anticoncentration and independence

4. Similar for smoothed analysis