MCA 2021

Congreso Matemático de las Américas
12-23 Julio

Mathematical Congress of the Americas
12-23 July

Buenos Aires Argentina

Symbolic vs Numerical: Allies or Enemies?

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& Josué Tonelli-Cueto (Inria Paris/IMJ-PRG)

Sesión especial / Special Session

Symbolic and Numerical Computation with Polynomials

Miércoles / Wednesday 14, 13:00-16:00
Viernes / Friday 16, 11:00-16:00
Miércoles / Wednesday 21, 16:00-21:00

Polynomials Everywhere I

Polynomials are used for modelling many phenomena:

applied

&

theoretical

Example 1: Chemical Reaction Networs

Two-component system
with hybrid histidine kinase

$$X_1 \xrightarrow{k_1} X_2 \xrightarrow{k_2} X_3 \xrightarrow{k_3} X_4$$

$$X_3 + X_5 \xrightarrow{k_4} X_1 + X_6$$

$$X_4 + X_5 \xrightarrow{k_5} X_2 + X_6$$

$$X_6 \xrightarrow{k_6} X_5$$

$\rightsquigarrow$

ODE of concentrations

$$\begin{cases} \dot{x}_1 = -k_1 x_1 + k_4 x_3 x_5 \\ \dot{x}_2 = k_1 x_1 - k_2 x_2 + k_5 x_4 x_5 \\ \dot{x}_3 = k_2 x_2 - k_3 x_3 - k_4 x_3 x_5 \\ \dot{x}_4 = k_3 x_3 - k_5 x_4 x_5 \\ \dot{x}_5 = -k_4 x_3 x_5 - k_5 x_4 x_5 + k_6 x_6 \\ \dot{x}_6 = k_4 x_3 x_5 + k_5 x_4 x_5 - k_6 x_6 \end{cases}$$

Example 1: Chemical Reaction Networs

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$$X_6 \xrightarrow{k_6} X_5$$

$\rightsquigarrow$

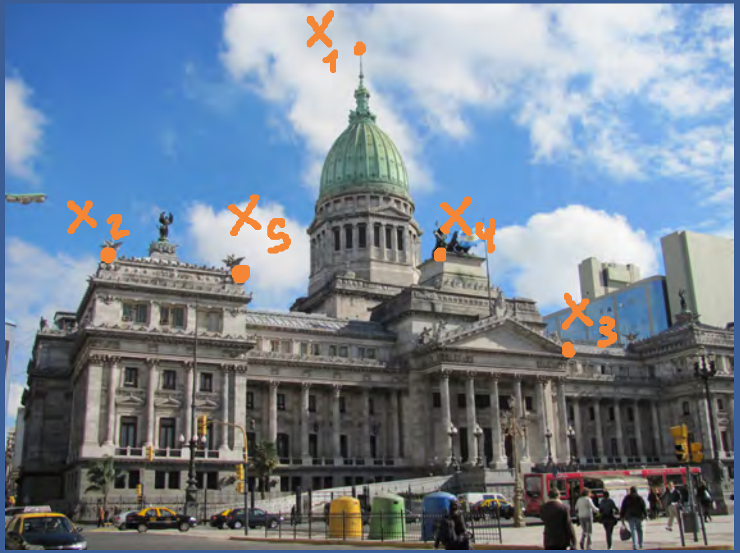
ODE of concentrations

$$\begin{cases} 0 = -k_1 x_1 + k_4 x_3 x_5 \\ 0 = k_1 x_1 - k_2 x_2 + k_5 x_4 x_5 \\ 0 = k_2 x_2 - k_3 x_3 - k_4 x_3 x_5 \\ 0 = k_3 x_3 - k_5 x_4 x_5 \\ 0 = -k_4 x_3 x_5 - k_5 x_4 x_5 + k_6 x_6 \\ 0 = k_4 x_3 x_5 + k_5 x_4 x_5 - k_6 x_6 \end{cases}$$

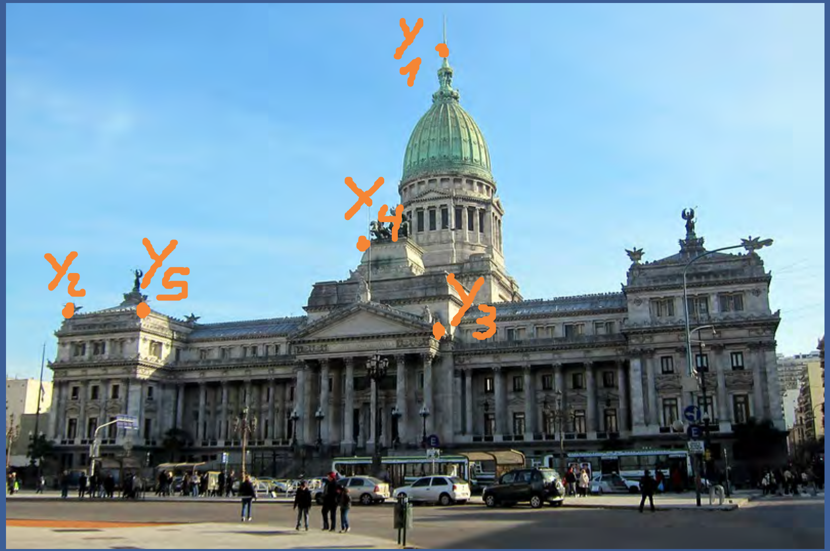
For which k_i we have more than one equilibrium (i.e. positive zero)?

More in: Bihan, Dickenstein & Giaroli. Lower bounds for positive roots and regions of multistationarity in chemical reaction networks.

Example 2. Algebraic Vision



X



Y

$(x_1, y_1) \in \mathbb{RP}^2 \times \mathbb{RP}^2$

$(x_2, y_2) \in \mathbb{RP}^2 \times \mathbb{RP}^2$

$(x_3, y_3) \in \mathbb{RP}^2 \times \mathbb{RP}^2$

$(x_4, y_4) \in \mathbb{RP}^2 \times \mathbb{RP}^2$

$(x_5, y_5) \in \mathbb{RP}^2 \times \mathbb{RP}^2$

Essential matrix: $E \in \mathbb{R}^{3\times 3}$ (gives relative orientations)

$$2EE^TE = \mathrm{tr}(EE^T)E \qquad \det E = 0 \qquad y_i^T E x_i = 0$$

More in: Nister. An efficient solution to the five-point relative pose problem.

POLYNOMIALS EVERYWHERE II

(Theoretical) Questions

Applications

Problems with polynomials

How to understand them?

How to solve them?

GEOMETRY OF POLYNOMIALS

Commutative algebra

Algebraic geometry

Differential geometry ...

A CASE STUDY FOR COMPUTATION: POLYNOMIAL SYSTEM SOLVING

HOW IS THE SYSTEM GIVEN?

- coefficients
  - $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{F}_p$, ...
  - exact or approximate?
- structured
  - dense
  - sparse

WHAT MEANS TO FIND A ZERO?

- Zeros in $\mathbb{C}^n$, $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\mathbb{Q}^n$, ...
- how to represent it?
  - approximation
    - à la Smale
    - isolating boxes
    - floating point
  - RUR
  - algebraic extensions (Gröbner bases)

HOW MUCH DOES COMPUTATION COST?

- Cost of operation
  - bit
  - arithmetic
- Complexity framework
  - worst-case
  - condition-based
  - probabilistic

WARNING:
NOT EVERYTHING IS FINDING ZEROS!

Existence of solutions

Elimination

Counting solutions

Implicitization

Computing topological invariants

Computing degree/dimension

SYMBOLIC vs. NUMERICAL

exact
vs.
approximate

input

output

operations

cost

bit vs. arithmetic

Complexity framework

worst-case

condition-based

probabilistic

implicit assumptions
of computation

A Roadmap to the Session

Existence of Solutions
Dickenstein
Sombra

Elimination/ Implicitization
D'Andrea Krick
Tsigaridas

Computing over p-adics
Rojas

Solving Structured Polynomial Systems
Busé Regan

Homotopy Continuation
Cucker Rodriguez
Malajovich Walker

Computing over $\mathbb{R}$
Szanto Yap

Numerical Complexity Theory
Burr Carrasco

Positive dimensional systems
Herrero Sottile
Ergür
Spheres: Etayo

Applications
Bürgisser Garrote Jeronimo

¡Muchas gracias por su atención!

And for the Europe-based researchers,

one advice for the session...

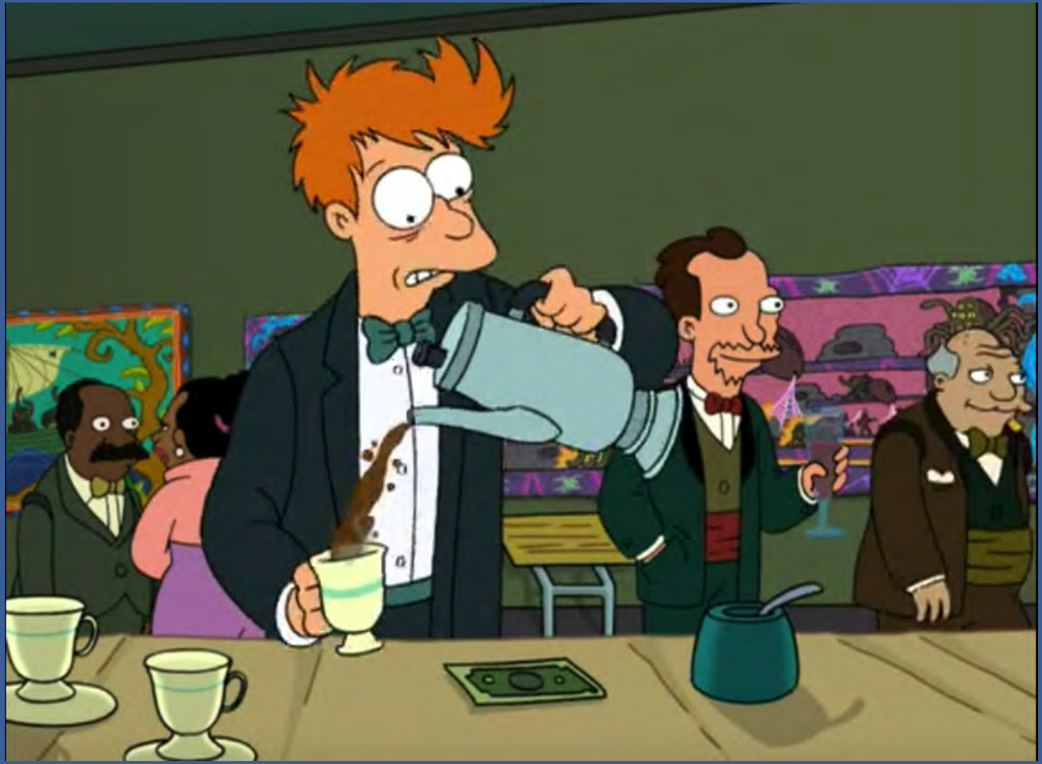


Image source: Futurama