Computing the Homology of Semialgebraic Sets via TDA

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Semialgebraic Sets

Formed from

atomic semialgebraic sets

i.e. sets described by

$(g=0), (g>0), (g\geq 0), (g<0), (g\leq 0)$

with g real polynomial

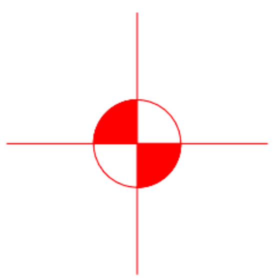
using **set-theoretical operations**

i.e. $\wedge$: intersection

$\vee$: union

$\neg$: complement

Example:



$((XY \leq 0) \wedge ((X^2+Y^2 \leq 1) \vee (XY=0))) \vee (X^2+Y^2=1)$

Why do we care?

Natural descriptions of many things are semialgebraic sets

Topological hardness

- Every finite simplicial complex is a semialgebraic set.
- (Gabrielov, Vorobjov; 2005, 2009)

$$\beta(S) \leq \mathcal{O}(q^2 D)^n$$

Main Result

THM There is a numerically stable algorithm that, given $g \in \mathbb{R}[X_1, \dots, X_n]^q$ with $\deg g \leq D$ and a semialg. formula Φ of size $\leq s$, computes $H_0(S(g,\Phi)), \dots, H_n(S(g,\Phi))$, where $S(g,\Phi)$ is the semialg. set described by g and Φ, in $s(qD)^{\mathcal{O}(n^3)}$-time with probability $\geq 1-(2qD)^{-n}$ when g is 'random'.

Outline of the algorithm

0) Homogeneization of g
1) Estimation cond. number of g, $\overline{\kappa}(g)$
2) Gabrielov-Vorobjov construction (general ineq → lax ineq) [Hard to make explicit!]
3) Create uniform grid for sample
4) Simplicial reconstruction of $S(g,\Phi)$: Construct simplicial model of $S(g,\Phi)$ by using Φ and simplicial model of atoms

Remarks

- All other algorithms have doubly exponential complexity in n
- Numerical algorithms $\Rightarrow$ $\begin{cases} \text{input-dependent run-time} \\ \text{possible ill-posed inputs} \\ \text{can handle errors} \end{cases}$
- Main TDA tool: Niyogi-Smale-Weinberger Thm
- (Cucker, Krick, Shub; 2018) Hardness of sampling dominated by cond. number of g in the algebraic case.

  (Bürgisser, Cucker, Lairez; 2018) Ext. to basic semialg. sets (no unions!)
- Unions require simplicial reconstruction!

Simplicial Reconstruction

Key observation (Non-formal)

- $\mathcal{X}_i$: sample of X_i

IF

$\forall I$, $\bigcap_{i\in I} \mathcal{X}_i$ 'good' sample of $\bigcap_{i \in I} X_i$

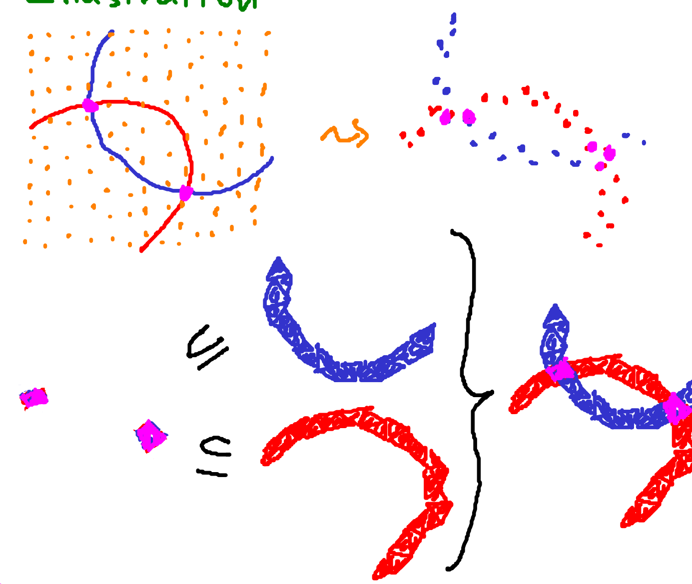
THEN

$\bigcup \check{C}_\varepsilon(\mathcal{X}_i)$ and $\bigcup X_i$: same homology

Remarks

- Proof uses a form of Vietoris-Begle with $\pi: \bigcup \check{C}_\varepsilon(\mathcal{X}_i) \to \bigcup \{[B_\varepsilon(y)] \mid y \in \bigcup \mathcal{X}_i\}$
- Valid also for Vietoris-Rips complex

Illustration



References

P. Bürgisser, F. Cucker, J. Tonelli-Cueto. Computing the Homology of Semialgebraic Sets. I: Lax Formulas & II. General Formulas. Found. Comp. Math., 2020 & 2021.
J. Tonelli-Cueto. Condition and Homology in Semialgebraic Geometry (Thesis).