UTSA
The University of Texas at San Antonio™

PROBLEM-SOLVING CLUB

created by Dr. Eduardo Dueñez in 2006

(Spring 2023)

Date & Time: Tuesday, 5:00 pm – 7:00 pm

Location: MH 3.03.10

Organizers: Christopher Duffer & Dr. Josué Tonelli-Cueto

Web: tonellicueto.xyz/UTSAProblemSolvingClub_Spring2023

0 General Problems

Problem 0.1. (UTSA Sandwich Problem) You eat sandwiches. You love sandwiches. No: you *love* sandwiches[1]. But *only* mini-sandwhiches that are exactly $2\,\text{in} \times 2\text{in} \times 1\text{in}$ in size[2]. When visiting Mexico last summer, you realized that, delicious as the tacos were south of the border, you *missed* your sandwiches, so naturally you asked Eduardo Dueñez to ship you some sandwiches from the UTSA Problem-Solving Club. He packed full and sent you a cubic box with an edge-length of 20 in—i.e. a total of 2000 mini-sandwiches! Given the urgency and sheer volume of the task, when it came to packing, all sandwiches ended up in random positions: some facing up or down, some facing either of the four sides of the box. When going through customs, an evil inspector took a very long needle and pierced right through the box (ouch!). Is there any chance that the needle may have missed every single sandwich (somehow finding its way right through their edges), or is it the case that necessarily some sandwich(es) must have been pierced by the needle?

Problem 0.2. (Rainbow Hats Puzzle) Seven prisoners are given the chance to be set free tomorrow. An executioner will put a hat on each prisoner's head. Each hat can be one of the seven colors of the rainbow and the hat colors are assigned completely at the executioner's discretion. Every prisoner can see the hat colors of the other six prisoners, but not his own. They cannot communicate with others in any form, or else they are immediately executed. Then each prisoner writes down his guess of his own hat color. If at least one prisoner correctly guesses the color of his hat, they all will be set free immediately; otherwise they will be executed. They are given the night to come up with a strategy. Is there a strategy that they can guarantee that they will be set free?

Problem 0.3. (Hades vs. Sisyphus) *(Spanish Mathematical Olympiad, 1993)* There are 1001 steps in a staircase. There are some rocks in some of the steps in such a way that there cannot be more than one rock per step. Sisyphus is fighting against Hades in this staircase by moving one rock alternatively. Each turn, Sisyphus has to take one rock and lift it, one or more steps, up to the first step that is empty. Then Hades takes one rock and brings it down to the step immediately inferior to it—which has to be empty, as no two rocks can be in the same step. In order to win, Sisyphus has to put a rock in the last step. Initially, there are 500 rocks in the initial 500 steps. If Sisyphus makes the first move, can Hades prevent Sisyphus to win?

Problem 0.4. (Gas in a Circuit) *(Spanish Mathematical Olympiad, 1997)* We have a circular circuit in which a car goes. The amount of gas needed for this car to go around the circuit is distributed in n deposits located in n fixed points of the circuit. At the beginning, there is no gas in the deposit of the car. Prove that no matter how the gas is distributed into the deposits and how the deposits are located we can always start at point in such a way that we can complete the circuit with the car. Assume that the consumption of gasoline is uniform and proportional to the distance moved. Moreover, assume

[1] Between 2018 and 2014, sandwiches—not pizza—were the snack choice at the UTSA Problem-Solving Club.

[2] Okay, truthfully, we like sandwiches of all shapes and sizes, but for the sake of this problem let's pretend we are very picky.

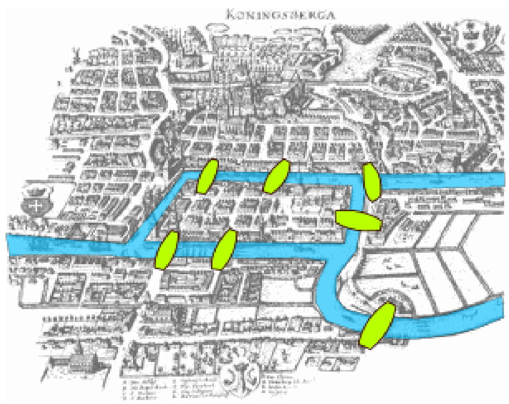
that no gas is wasted when we stop to fill the deposit of the gas at a deposit and that the deposit of the car can hold all the gas needed to go around once.

1 Parity Principle

Problem 1.1. In a party, there are 2006 guest divided across four rooms. Prove that it is always possible to join the guest of the first room with the guests of another room in such a way that they can form couples for dancing without leaving anyone out.

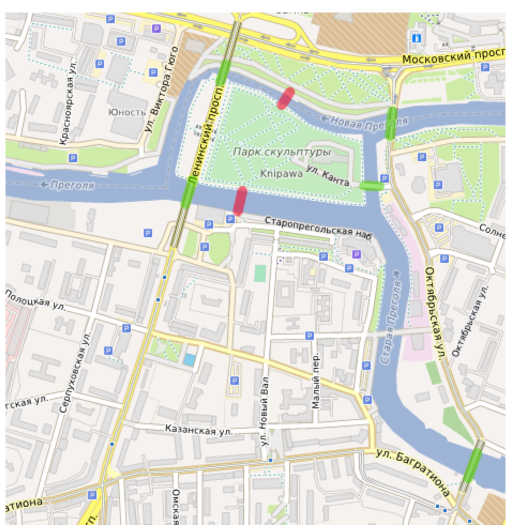
Problem 1.2. Can an ant walk through all edges of a cube in such a way that it does not pass twice through the same edge?

Problem 1.3. **(Euler's Königsberg Bridge Problem)** The following map of city of Königsberg emphasizes the river and bridges of the city.



A common pass-time at the time was trying to walk through the city passing through all the bridges once and only once. However, no one was able to achieve it. Can you explain why?

Nowadays, the map of of Königsberg (now called Kaliningrad) is the following one:



In this map, we can see in red the bridges that have been destroyed from Euler's times; in green, the ones that have survived; and in white, the new ones. Can we go through every bridge crossing each bridge only once now?

And finally, is this possible in San Antonio?

Problem 1.4. A battlefield is divided into 64 sectors through a 8×8 grid. The objective of the attacking army is hidden in one of the sector and for trying to achieve it they throw their soldiers in parachute from the air. Assume that the sector in which a paratrooper falls is fully random and that a paratroopers cannot fall in a sector in which there is already a paratrooper. Once in land, the soldiers can move two sectors at a time but they cannot stop on the way to explore the adjacent sectors. Which is the minimum number of soldiers that the army should send to find the objective certainly and successfully? Does anything change if the army can choose where the soldiers land?

Problem 1.5. In a board formed by an unique row of 2006 squares, there is one token in the first square and one piece in the last square. The tokens have two sides: one black and one white. The first token shows the white side and the last token the black one. We can move the tokens from square to square in any direction (one square at a time), but each time we move a token we flip it around. Moreover, in an square, two tokens can come together only if they disagree on color. Is it possible to move the token in the last square to the first square and the token in the last square to the first square?

Problem 1.6. Is it possible to place 31 domino pieces forming a square in which the two opposite corners are missing? And if the corners don't have to be opposite to each other?

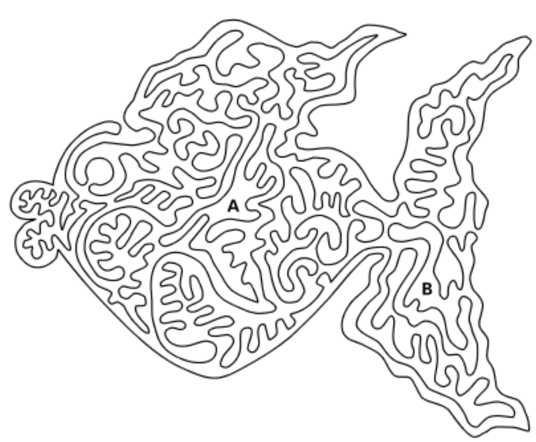
Problem 1.7. We put 13 points $P_1, P_2, \ldots, P_{13}$ in the plane and we draw the segments P_1P_2, P_2P_3, P_3P_4,. . .,$P_{12}P_{13}$, $P_{13}P_1$. Is it possible to draw a straight line in such a way that it cuts all segments in their interior (without touching the extremes)? And if instead of a straight line we draw a circle?

Problem 1.8. Let a_1, . . ., a_{2007} be the first 2007 positive whole numbers in some order. Prove that the number

$$(a_1 - 1)(a_2 - 2)(a_3 - 3)\cdots(a_{2007} - 2007)$$

is even.

Problem 1.9. Is it possible to go from point A to point B in the drawing below without crossing the walls?



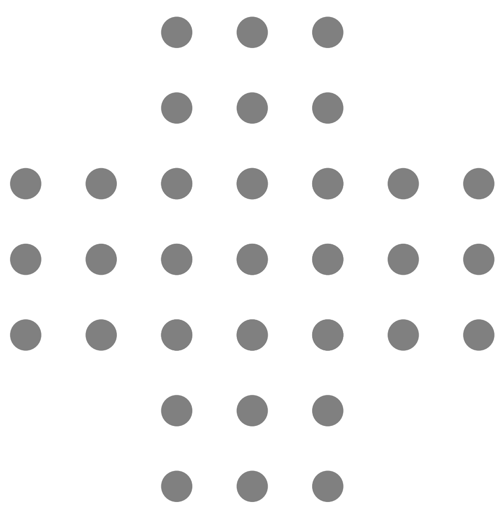
Problem 1.10. We put 40 cards face down forming 5 rows with 8 cards per row. Two players play a game as follows: In each turn, a player picks a row where not all the cards are face up and turns face up a card that is face down; then the player can turn any of the cards that follow in the row however they want—for each of the cards that follow, either E flips it or E leave leaves it as it is. The first player who cannot do any movement (because all cards are face up) loses the game. Prove that the first player can always win.

Problem 1.11. (Nim game) In the *nim game* we start with five piles of 1, 2, 3, 4 and 5 objects, respectively. Two players make alternatively the following move: they select a pile in which objects remain and remove from it any (non-zero) amount of objects they desire. The first player who cannot make any move loses, i.e., the player who leaves no objects into play at the end of their turn wins. Prove that the first player can always win. What happens if we increase the number of piles or vary the number of objects in each pile?

Problem 1.12. (Hats in a prison) In a prison, 100 prisoners are standing in a queue facing in one direction. Each prisoner is wearing either an orange hat or a blue hat. A prisoner can see the hats of all the prisoners in front of Em in the queue, but can see neither Eir own hat nor the hats of the prisoners standing behind. The warden of the prison is going to ask each prisoner for the color Eir hat starting from the last prisoner in queue. If a prisoner guesses the color of Eir hat, then E is liberated, otherwise E is executed. How many prisoners can be saved at most if the warden allows them to come up with an strategy before he starts asking?

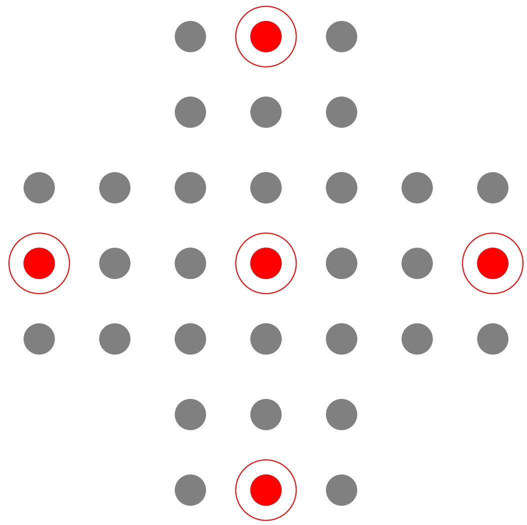
Problem 1.13. *(Basque Mathematical Olympiad, 2004)* In a chessboard—i.e., an 8×8 board—we put 24 pieces in the first three top rows. We can move the pieces as follows: a piece can jump over any other piece to a free position horizontally, vertically or diagonally. Can we put all the pieces in the bottom three rows?

Problem 1.14. (Peg Solitaire) In the English *peg solitaire*, we start with a board with the following shape:

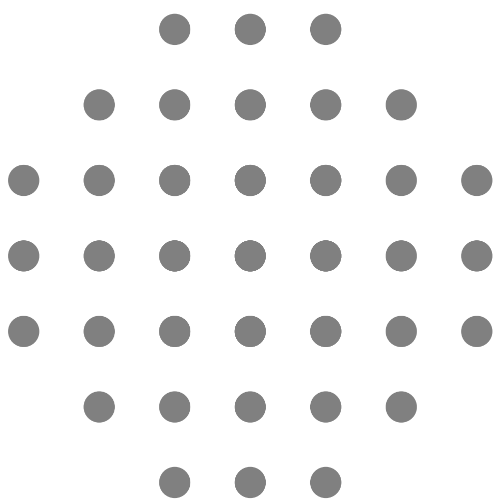


In words, it is a board with 33 positions distributed across two 3×7 rectangles that cross perpendicularly in a 3 square. In this board, all positions except the central one are occupied by 32 pieces.

We can move the pieces as follows: a piece can jump over any other piece to a free position horizontally and vertically, but then the piece it jumped over is removed from the boards. The objective of this one-person game is to remove all pieces except one (which cannot be moved or removed, as no valid movement would be available then) leaving this remaining piece in the center of the board. Prove[1] that if only one piece remains, then this piece must lie in one of the following positions: the central position, or one of the four extreme positions which are 3 place up, down, left and right from the center. These positions are marked in the image below:



In the European version of *peg solitaire*, the board is slightly different having 37 positions: we add a position in each of the inwards corners of the cross as the picture below shows:



The rules and objective of the one-person game are the same. Now, if only one piece remains, in which positions should this piece end?

[1] **Hint**: Color each position with orange and blue in such a way that every three consecutive positions, vertically or horizontally, have two two orange positions and one blue position.

And what about boards with other shapes that are symmetric under right angle rotations around the central position?

Problem 1.15. *Spanish Mathematical Olympiad, 2004* We put in a circle 2004 bi-sided tokens which are orange on one side and blue on the other. We can perform the following movement: turn around any token together with its two adjacent tokens. If initially there is only one single piece with the orange side upwards, can we make all tokens blue repeating the described movement? And, if instead of 2004 tokens, we started with 2003, would it be possible then?

Problem 1.16. We have an 8×8 grid with 64 bulbs. In each row and column, we have a switch that when switched changes the state of all the bulbs in the corresponding row or column, i.e., the bulbs that were on are turned off, and the ones that were off are turned on. If no bulb is on at the beginning, can we switch the switches around in such a way that we end with only one bulb on—no matter where—at the end? And what about getting only all the bulbs in one of the diagonals on?

Bibliography

[1] *Mathematics Stack Exchange*. `https://math.stackexchange.com/`.

[2] J. Sangroniz Gómez. *Los problemas de ingenio como recurso didáctico para las matemáticas en la enseñanza secundaria*. Programa Garatu 2006/2007. 2006.